EQUIVARIANT COHOMOLOGY AND THE MAURER-CARTAN EQUATION

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Abstract. Let G be a compact, connected Lie group, acting smoothly on a manifold M. In their 1998 paper, Goresky-Kottwitz-MacPherson described a small Cartan model for the equivariant cohomology of M, quasi-isomorphic to the standard (large) Cartan complex of equivariant differential forms. In this paper, we construct an explicit cochain map from the small Cartan model into the large Cartan model, intertwining the $(S\mathfrak{g}^*)_{\mathrm{inv}}$-module structures and inducing an isomorphism in cohomology. The construction involves the solution of a remarkable inhomogeneous Maurer-Cartan equation. This solution has further applications to the theory of transgression in the Weil algebra, and to the Chevalley-Koszul theory of the cohomology of principal bundles.

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Contents

1. Introduction

Let G be a compact connected Lie group of rank l, and $\pi\colon P \to B$ a principal G-bundle with connection. Choose a connection on P. By the Chern-Weil construction, the de Rham complex $\Omega(B)$ of differential forms on the base becomes a module for the algebra $(S\mathfrak{g}^*)_{\mathrm{inv}}$ of invariant polynomials. Consider the corresponding Koszul complex,

$$\Omega(B)\otimes(\wedge\mathfrak{g}^*)_{\mathrm{inv}},\quad \mathrm{d}\otimes 1+\sum_j p^j\otimes\iota(c_j), \tag{1}$$

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where $c_1,\ldots,c_l$ are the primitive generators of $(\wedge\mathfrak{g})_{\mathrm{inv}}$, and $p^1,\ldots,p^l$ are the generators of $(S\mathfrak{g}^*)_{\mathrm{inv}}$, corresponding to the dual basis $c^j\in(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ by Chevalley's transgression theorem. It is a classical result of Chevalley and Koszul [15, 7], that the complex (1) is quasi-isomorphic to the complex $\Omega(P)_{\mathrm{inv}}$ of invariant forms on the total space.

Goresky-Kottwitz-MacPherson in [6] described a similar "small" model for the equivariant de Rham cohomology of any G-manifold M. Recall that the standard Cartan model for the equivariant de Rham cohomology of M is the complex

$$(S\mathfrak{g}^*\otimes\Omega(M))_{\mathrm{inv}},\ 1\otimes\mathrm{d}-\sum_a v^a\otimes\iota(e_a), \tag{2}$$

where $e_a\in\mathfrak{g}$ is a basis, and $v^a\in S\mathfrak{g}^*$ are the generators of the symmetric algebra given by the dual basis. By contrast, the small Cartan model introduced in [6] involves only *invariant* differential forms:

$$(S\mathfrak{g}^*)_{\mathrm{inv}}\otimes\Omega(M)_{\mathrm{inv}},\quad 1\otimes\mathrm{d}-\sum_j p^j\otimes\iota(c_j). \tag{3}$$

The goal of the present paper is the construction of an explicit cochain map from the small Cartan model (3) into the Cartan model (2), commuting with the $(S\mathfrak{g}^*)_{\mathrm{inv}}$-module structure and inducing an isomorphism in cohomology. In more detail, we will construct a nilpotent even element

$$f\in(S\mathfrak{g}^*\otimes\wedge\mathfrak{g})_{\mathrm{inv}},$$

such that the natural inclusion from (3) to (2), followed by a 'twist' by the operator $e^{\iota(f)}$ (letting $\wedge\mathfrak{g}$ act by contraction), is the desired cochain map. As it turns out, these properties are equivalent to the following Maurer-Cartan equation for the element f,

$$\partial f+\tfrac{1}{2}[f,f]_{\wedge\mathfrak{g}}=\sum_j p^j\otimes c_j-\sum_a v^a\otimes e_a. \tag{4}$$

Here ∂ is the Lie algebra boundary operator on $\wedge\mathfrak{g}$, and $[\cdot,\cdot]_{\wedge\mathfrak{g}}$ the Schouten bracket, both extended to the algebra $S\mathfrak{g}^*\otimes\wedge\mathfrak{g}$ in the natural way. The main discovery of this paper is that this equation does, in fact, have a solution.

The original argument in [6] for the equivalence of the two Cartan models was based on the Koszul duality between differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules and $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules. In this approach, one has to show that (1) is quasi-isomorphic to $\Omega(P)_{\mathrm{inv}}$ not just as a differential space, but also as a differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-module. As pointed out in Mazszyk-Weber [17], the proof of this fact in [6] contains an error (the proposed map is not a cochain map) . However, the alternative argument in [17] is incorrect as well (the cochain map given there does not respect the $(\wedge\mathfrak{g})_{\mathrm{inv}}$-module structure). As we will explain in this paper, the desired quasi-isomorphism of differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules may be constructed by once again employing the twist $e^{\iota(f)}$.

The element f also provides a new point of view on transgression. Let $W\mathfrak{g}=S\mathfrak{g}^*\otimes\wedge\mathfrak{g}^*$ be the Weil algebra equipped with the Weil differential d^W. Then, $e^{\iota(f)}$ is an operator acting on $W\mathfrak{g}$. We show that for any primitive element $c^j\in(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ one has

$$\mathrm{d}^W\left(e^{\iota(f)}(1\otimes c^j)\right)=p^j\otimes 1.$$

That is, $e^{\iota(f)}(1\otimes c^j)$ is a cochain of transgression for the polynomial p^j.

We would like to point out some recent references related to this work. In [1], Allday-Puppe prove a version of a conjecture of Goresky-Kottwitz-MacPherson that the small Cartan model may be replaced with an even smaller 'Hirsch-Brown' model, $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes H(M)$, where the differential is constructed from cohomology operations over M. In a different direction, Franz [5, 4] has introduced small Cartan-type models for the equivariant cohomology with *integer* coefficients. Huebschmann [10, 11] obtained small Cartan models using homological perturbation techniques.

The organization of the paper is as follows. In Section 2, we collect some formulas for the Lie algebra boundary and coboundary operators, and review Chevalley's theory of transgression in the Weil algebra. In Section 3 we consider the problem of finding a cochain map from the Koszul algebra over the space of primitive elements into the Weil algebra. This naturally leads to the above Maurer-Cartan equation. We prove that the Maurer-Cartan equation admits a solution, which is unique up to 'gauge transformations'. In the subsequent Sections 4 and 5 we apply our results to the small Cartan and the Chevalley-Koszul complexes. Finally, in the appendix we explain how the two complexes may be viewed as special cases of a more general complex due to Halperin [7], and how they are related by Koszul duality [6].

Throughout, we will work in the algebraic context of differential spaces, over any field $\mathbb{F}$ of characteristic zero. The applications to manifolds are obtained as special cases for $\mathbb{F} = \mathbb{R}$, working with complexes of differential forms.

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2. Preliminaries

In this Section we recall some basic results (due largely to Chevalley, Hopf, and Koszul) concerning the structure of the invariant subspace of the symmetric and exterior algebra over any reductive Lie algebra. For more details, see [14, 13, 7].

2.1. **Graded vector spaces.** Throughout this paper, $\mathbb{F}$ will denote a field of characteristic 0. We will frequently encounter graded vector spaces $V = \bigoplus_{i\in\mathbb{Z}} V^i$ over $\mathbb{F}$. Such vector spaces form a category $\mathrm{GR}_{\mathbb{F}}$, with morphisms the linear maps preserving degree. Given a graded vector space V, we denote by $V[k]$ the vector space V with the shifted grading $V[k]^i = V^{k+i}$. A linear map $V \to W$ between graded vector spaces has degree k if it defines a morphism $V \to W[k]$. The tensor product $V\otimes W$ of two graded vector spaces carries a grading $(V\otimes W)^i = \bigoplus_{r+s=i} V^r \otimes W^s$. Define the commutativity isomorphism $V \otimes W \to W \otimes V$ by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, where $|\cdot|$ denote the degree of a homogeneous element. Together with the obvious associativity isomorphism, $U \otimes (V \otimes W) \to (U \otimes V) \otimes W$, this makes $\mathrm{GR}_{\mathbb{F}}$ into a tensor category. One can therefore consider its algebra objects (called graded algebras), Lie algebra objects (called graded Lie algebras), and so forth. The commutativity isomorphism encodes the *super sign convention*: For instance, if $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i$ is a graded algebra, we denote by $[\cdot,\cdot]$ the (graded) commutator

$$[x, y] = xy - (-1)^{|x||y|} yx.$$

(This makes $\mathcal{A}$ into a graded Lie algebra; $\mathcal{A}$ is called commutative if the bracket is trivial.) A derivation of $\mathcal{A}$ is a linear map $\partial \in \mathrm{End}(\mathcal{A})$ such that

$$[\partial, \epsilon(x)] = \epsilon(\partial x)$$

where $\epsilon\colon \mathcal{A} \to \mathrm{End}(\mathcal{A})$ is given by left multiplication. (Later, we will usually omit ϵ from the notation.) Similarly, one defines derivations of graded Lie algebras.

2.2. **Lie algebra homology and cohomology.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. In order to avoid confusion with commutators, the Lie bracket will be denoted $[\cdot,\cdot]_{\mathfrak{g}}$. For any $\mathfrak{g}$-module $\mathcal{M}$, the operator corresponding to $\xi \in \mathfrak{g}$ is denoted $L^{\mathcal{M}}(\xi)$, or simply $L(\xi)$ if $\mathcal{M}$ is clear from the context.

Consider the exterior powers of the adjoint and coadjoint representations, with gradings

$$(\wedge\mathfrak{g})^{-i} = \wedge^i\mathfrak{g}, \quad (\wedge\mathfrak{g}^*)^i = \wedge^i\mathfrak{g}^*.$$

For $\xi \in \mathfrak{g}$ we denote by

$$\epsilon(\xi) \in \mathrm{End}(\wedge\mathfrak{g}), \ \iota(\xi) \in \mathrm{End}(\wedge\mathfrak{g}^*)$$

the operators of exterior multiplication and contraction. Note that both of these operators have degree -1, hence they extend to homomorphisms of graded algebras,

$$\epsilon\colon \wedge\mathfrak{g} \to \mathrm{End}(\wedge\mathfrak{g}), \quad \iota\colon \wedge\mathfrak{g} \to \mathrm{End}(\wedge\mathfrak{g}^*). \tag{5}$$

Dually, for $\mu \in \mathfrak{g}^*$ we define operators of degree $+1$, $\iota^*(\mu) \in \mathrm{End}(\wedge\mathfrak{g})$, $\epsilon^*(\mu) \in \mathrm{End}(\wedge\mathfrak{g}^*)$. Recall Koszul's formulas for the Lie algebra differentials $\mathrm{d} \in \mathrm{End}(\wedge\mathfrak{g}^*)$ and $\partial \in \mathrm{End}(\wedge\mathfrak{g})$,

$$\mathrm{d} = \tfrac{1}{2}\sum_a \epsilon^*(e^a)L(e_a), \quad \partial = -\tfrac{1}{2}\sum_a L(e_a)\iota^*(e^a) \tag{6}$$

where $e_a \in \mathfrak{g}$ and $e^a \in \mathfrak{g}^*$ are dual bases. Both of these operators square to 0 and are $\mathfrak{g}$-equivariant. The differential d is a derivation of $\wedge\mathfrak{g}^*$, while ∂ is a coderivation of the natural coproduct on $\wedge\mathfrak{g}$. On the other hand, the interaction of ∂ with the product on $\wedge\mathfrak{g}$ is given by the formula (cf. [7, p.178])

$$\partial(f \wedge g) = \partial f \wedge g + (-1)^{|f|} f \wedge \partial g + (-1)^{|f|}[f,g]_{\wedge\mathfrak{g}} \tag{7}$$

where $[\cdot,\cdot]_{\wedge\mathfrak{g}}$ is the *Schouten bracket*,

$$[f,g]_{\wedge\mathfrak{g}} = -\sum_a L(e_a)f \wedge \iota^*(e^a)g. \tag{8}$$

The Schouten bracket makes $(\wedge\mathfrak{g})[1]$ into a graded Lie algebra, with $\mathfrak{g}$ as a Lie subalgebra. The differential ∂ is a derivation of the bracket, so that $(\wedge\mathfrak{g})[1]$ is a differential graded Lie algebra. The center of $(\wedge\mathfrak{g})[1]$ is the invariant subspace $(\wedge\mathfrak{g})_{\mathrm{inv}}[1]$, with the zero differential.

We will need the following generalization of Cartan's formula $[\mathrm{d}, \iota(\xi)] = L(\xi)$ for $\xi \in \mathfrak{g}$:

Lemma 2.1. *For any $f \in \wedge\mathfrak{g}$,*

$$[\mathrm{d}, \iota(f)] = -\iota(\partial f) + \sum_a \iota(\iota^*(e^a)f)\, L(e_a). \tag{9}$$

Proof. The proof is by induction on the degree of f, the case $|f| = 1$ being Cartan's formula. Suppose $f = \xi \wedge g$ where $|g| = |f| - 1$ and $\xi \in \mathfrak{g}$. By induction, we may assume that the formula holds for g. Thus

$$\begin{aligned}[\mathrm{d}, \iota(\xi \wedge g)] &= L(\xi)\iota(g) - \iota(\xi)[\mathrm{d}, \iota(g)] \\ &= \iota(L(\xi)g) + \iota(g)L(\xi) + \iota(\xi)\iota(\partial g) - \sum_a \iota(\xi)\iota(\iota^*(e^a)g)L(e_a).\end{aligned}$$

The second term can be written as $\iota(\iota^*(e^a)\xi \wedge g)L(e_a)$, which combines with the fourth term to $\sum_a \iota(\iota^*(e^a)f)L(e_a)$. The first and third term add to $-\iota(\partial f)$ since $\partial(\xi \wedge g) = -\xi \wedge \partial g - L(\xi)g$. □

Let $(\wedge \mathfrak{g})^- = \bigoplus_{i>0} \wedge^i \mathfrak{g}$. Since any $f \in (\wedge \mathfrak{g})^-_{\mathrm{even}}$ is nilpotent, the exponential $e^f = \sum_{n=0}^\infty \frac{1}{n!} f^n$ is given by a finite series.

Lemma 2.2. *For any $f \in (\wedge \mathfrak{g})^-_{\mathrm{even}}$,*

$$e^{-\iota(f)} \circ \mathrm{d} \circ e^{\iota(f)} = \mathrm{d} - \iota(\partial f + \tfrac{1}{2}[f, f]_{\wedge \mathfrak{g}}) + \sum_a \iota(\iota^*(e^a)f)L(e_a). \tag{10}$$

Proof. We may write $e^{-\iota(f)} \circ \mathrm{d} \circ e^{\iota(f)}$ as a sum,

$$\mathrm{Ad}(e^{-\iota(f)})\mathrm{d} = \mathrm{d} + [\mathrm{d}, \iota(f)] + \tfrac{1}{2}\big[[\mathrm{d}, \iota(f)], \iota(f)\big] + \cdots .$$

The first commutator $[\mathrm{d}, \iota(f)]$ was computed in (9). The next commutator is

$$\big[[\mathrm{d}, \iota(f)], \iota(f)\big] = \sum_a \iota\big(\iota^*(e^a)f \wedge L(e_a)f\big) = -\iota([f, f]_{\wedge \mathfrak{g}}),$$

and all higher commutators vanish. □

Lemma 2.3. *For any $f \in (\wedge \mathfrak{g})^-_{\mathrm{even}}$,*

$$e^{-f}\partial(e^f) = \partial f + \tfrac{1}{2}[f, f]_{\wedge \mathfrak{g}}. \tag{11}$$

Proof. Applying (9) to the element e^f, we find

$$\begin{aligned}\mathrm{d} \circ \iota(e^f) &= \iota(e^f) \circ \mathrm{d} - \iota(\partial e^f) + \sum_a \iota(\iota^*(e^a)e^f)L(e_a) \\ &= \iota(e^f) \circ \big(\mathrm{d} - \iota(e^{-f}\partial e^f) + \sum_a \iota(\iota^*(e^a)f)L(e_a)\big).\end{aligned}$$

Equation (11) follows by comparing this formula with (10). □

Below we will use these formulas in slightly greater generality: Suppose $\mathcal{B}$ is a commutative, evenly graded algebra. We use the same notation for the contraction operation (5) , the differentials (6), and the Schouten bracket (8), and for their extensions to $\mathcal{B} \otimes \wedge \mathfrak{g}$, $\mathcal{B} \otimes \wedge \mathfrak{g}^*$, by $\mathcal{B}$-linearity. Then Equations (10) and (11) hold for any even element $f \in \mathcal{B} \otimes (\wedge \mathfrak{g})^-$, by the same proof.

2.3. **Hodge theory on $\wedge\mathfrak{g}$.** Suppose now that $\mathfrak{g}$ is a reductive Lie algebra. Then the projection onto invariants $\wedge\mathfrak{g} \to (\wedge\mathfrak{g})_{\mathrm{inv}}$ is a homotopy equivalence for the differential ∂, with homotopy inverse the inclusion [7, p.189]. Recall the construction of a homotopy operator, using Hodge theory with respect to an invariant scalar product B on $\mathfrak{g}$. Let $B^\flat\colon \mathfrak{g} \to \mathfrak{g}^*$ and $B^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$ denote the isomorphisms defined by B, and let

$$\mathrm{Cas}_{\mathfrak{g}} = \sum_a B^\sharp(e^a) e_a \in U(\mathfrak{g})_{\mathrm{inv}}$$

be the quadratic Casimir operator. Let $\mathrm{Cas}_{\mathfrak{g}}^\wedge$ denote the operator on $\wedge\mathfrak{g}$ defined by $\mathrm{Cas}_{\mathfrak{g}}$ via the adjoint representation. Then $\wedge\mathfrak{g} = \mathrm{im}\,\mathrm{Cas}_{\mathfrak{g}}^\wedge \oplus \ker \mathrm{Cas}_{\mathfrak{g}}^\wedge$, and $\ker \mathrm{Cas}_{\mathfrak{g}}^\wedge$ is the invariant subspace $(\wedge\mathfrak{g})_{\mathrm{inv}}$. The isomorphisms $B^\flat$ and $B^\sharp$ extend to the exterior algebras, and in particular the Lie algebra differential d on $\wedge\mathfrak{g}^*$ defines a differential on $\wedge\mathfrak{g}$

$$\delta = B^\sharp \circ \mathrm{d} \circ B^\flat \in \mathrm{Der}(\wedge\mathfrak{g}).$$

The corresponding Hodge Laplacian

$$\mathcal{L} = \delta\partial + \partial\delta \in \mathrm{End}(\wedge\mathfrak{g})$$

has degree 0, and equals $-\frac{1}{2}\mathrm{Cas}_{\mathfrak{g}}^\wedge$ [13, Equation (94)]. Hodge theory shows that $\mathrm{im}\,\mathcal{L} = \mathrm{im}\,\delta \oplus \mathrm{im}\,\partial$, and one obtains the direct sum decomposition [13, Proposition 22],

$$\wedge\mathfrak{g} = \mathrm{im}\,\delta \oplus \mathrm{im}\,\partial \oplus (\wedge\mathfrak{g})_{\mathrm{inv}}. \tag{12}$$

Let $\mathcal{G} \in \mathrm{End}(\wedge\mathfrak{g})$ denote the Green's operator, i.e. $\ker\mathcal{G} = \ker\mathcal{L}$ and $\mathcal{L}\mathcal{G} = \mathcal{G}\mathcal{L} = I - \Pi$ where $\Pi\colon \wedge\mathfrak{g} \to (\wedge\mathfrak{g})_{\mathrm{inv}}$ is the projection defined by the splitting. Then $\mathcal{G}$ has degree 0, and $\mathcal{S} = \mathcal{G}\delta = \delta\mathcal{G}$ is the desired homotopy operator:

$$[\mathcal{S}, \partial] = [\mathcal{G}\delta, \partial] = \mathcal{G}[\delta, \partial] = \mathcal{G}\mathcal{L} = I - \Pi.$$

Using that $\mathfrak{g}$ is reductive, the differential ∂ may be written $\partial = -\frac{1}{2}\sum_a \iota^*(e^a)\ L(e_a)$, and in particular vanishes on invariants. Hence one obtains an isomorphism of vector spaces, $H(\wedge\mathfrak{g}, \partial) = (\wedge\mathfrak{g})_{\mathrm{inv}}$. Similarly, the inclusion $(\wedge\mathfrak{g}^*)_{\mathrm{inv}} \hookrightarrow \wedge\mathfrak{g}^*$ defines an isomorphism of algebras $H(\wedge\mathfrak{g}^*, \mathrm{d}) = (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$.

2.4. **Primitive elements.** Since $\mathfrak{g}$ is reductive, the pairing between $\wedge\mathfrak{g}$ and $\wedge\mathfrak{g}^*$ restricts to a non-degenerate pairing between $(\wedge\mathfrak{g})_{\mathrm{inv}}$ and $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. As a consequence, the algebra structure on $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ induces a coalgebra structure on $(\wedge\mathfrak{g})_{\mathrm{inv}}$, and the algebra structure on $(\wedge\mathfrak{g})_{\mathrm{inv}}$ induces a coproduct on $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. The coproduct and product are compatible in both cases, turning $(\wedge\mathfrak{g})_{\mathrm{inv}}$ and $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ into commutative graded Hopf algebras. Recall that an element x of a graded coalgebra is called *primitive* if it has the property,

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

where Δ is the coproduct. Let $\mathcal{P}, \mathcal{P}^*$ denote the graded subspaces of primitive elements in $(\wedge\mathfrak{g})_{\mathrm{inv}}, (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$, respectively. It can be shown [7, page 206] that the pairing between $(\wedge\mathfrak{g})_{\mathrm{inv}}, (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ restricts to a non-degenerate pairing between $\mathcal{P}$ and $\mathcal{P}^*$, so that indeed $\mathcal{P}^*$ is the dual space to $\mathcal{P}$. By results of Hopf and Samelson, the elements in $\mathcal{P}, \mathcal{P}^*$ all have odd degree, and the inclusion maps extend to graded Hopf algebra isomorphisms

$$\wedge\mathcal{P} \xrightarrow{\cong} (\wedge\mathfrak{g})_{\mathrm{inv}}, \quad \wedge\mathcal{P}^* \xrightarrow{\cong} (\wedge\mathfrak{g}^*)_{\mathrm{inv}}.$$

This means in particular that the operator of contraction by any $c \in \mathcal{P}$ is a *derivation* of $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$, even though it is not of course a derivation of $\wedge\mathfrak{g}^*$.

2.5. **Transgression.** Let $S\mathfrak{g}^*$ be the symmetric algebra over $\mathfrak{g}^*$, with grading

$$(S\mathfrak{g}^*)^{2i} = S^i\mathfrak{g}^*, \quad (S\mathfrak{g}^*)^{2i+1} = 0.$$

Let $\tilde{\mathcal{P}} = \mathcal{P}[-1]$ be the evenly graded vector space, obtained by lowering the grading of $\mathcal{P}$ by 1, and dually $\tilde{\mathcal{P}}^* = \mathcal{P}^*[1]$. There is a canonical isomorphism of graded algebras, due to Koszul and Chevalley (see [7, page 242])

$$S\tilde{\mathcal{P}}^* \cong (S\mathfrak{g}^*)_{\mathrm{inv}}.$$

Let us review Chevalley's construction [7, page 363] of this isomorphism, using transgression in the Weil algebra

$$W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*.$$

Fix a basis e_a of $\mathfrak{g}$, with dual basis e^a, and let $y^a \in \wedge^1\mathfrak{g}^*$ and $v^a \in S^1\mathfrak{g}^*$ be the corresponding generators of the exterior and symmetric algebra. The Weil differential d^W is the derivation of $W\mathfrak{g}$ given by the formula, [1]

$$\mathrm{d}^W = \sum_a y^a L^W(e_a) - \mathrm{d}^\wedge + \sum_a v^a \iota(e_a). \tag{13}$$

Here $L^W(e_a) = L^S(e_a) + L^\wedge(e_a)$ are the generators for the $\mathfrak{g}$-action, while $\mathrm{d}^\wedge, \iota$ are the Lie algebra differential and the contraction operator, acting on the second factor, $\wedge\mathfrak{g}^*$. The Weil algebra is acyclic, and so is the invariant subalgebra $(W\mathfrak{g})_{\mathrm{inv}}$. The subspace $(S\mathfrak{g}^*)_{\mathrm{inv}} \subset W\mathfrak{g}$ consists of cocycles for the Weil differential. Hence, by acyclicity, any element in $(S^+\mathfrak{g}^*)_{\mathrm{inv}}$ is exact.

An odd element $x \in (W\mathfrak{g})_{\mathrm{inv}}$ is called a *cochain of transgression* if its differential $\mathrm{d}^W x$ lies in $(S\mathfrak{g}^*)_{\mathrm{inv}}$. It is called a *distinguished cochain of transgression* if, in addition, $\iota(c)x \in \mathbb{F}$ for all $c \in P$. The space of cochains of transgression is denoted $\mathcal{T}$, and its subspace of distinguished cochains of transgression $\mathcal{T}_{\mathrm{dist}}$. The image of $\mathcal{T} \subset W\mathfrak{g}$ under the projection $W\mathfrak{g} \to \wedge\mathfrak{g}^*$ (defined by the augmentation map $S\mathfrak{g}^* \to \mathbb{F}$) is exactly the subspace $\mathcal{P}^* \subset (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ of primitive elements. Together with the map $\mathcal{T} \to (S\mathfrak{g}^*)_{\mathrm{inv}}$, $x \mapsto \mathrm{d}^W x$ this fits into a commutative diagram,

$$\begin{tikzcd} & \mathcal{T} \arrow[dl] \arrow[d] & \\ (S^+\mathfrak{g}^*)_{\mathrm{inv}} \arrow[r, "\tau"'] & \mathcal{P}^* & \subset (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \end{tikzcd}$$

The map $\tau\colon (S^+\mathfrak{g}^*)_{\mathrm{inv}} \to \mathcal{P}^*$ has degree -1 and is referred to as *transgression.* Its kernel is the annihilator of the ideal $\big((S^+\mathfrak{g})_{\mathrm{inv}}\big)^2 \subset (S^+\mathfrak{g})_{\mathrm{inv}}$ of decomposable elements. As it turns out [7, page 239], the map from $\mathcal{T}_{\mathrm{dist}}$ to $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ is still onto $\mathcal{P}^*$, and there is a *unique* map

[1] From now on, we identify the elements of any algebra with the corresponding operator of left multiplication on the algebra.

$\gamma\colon \mathcal{P}^* \to (S\mathfrak{g}^*)_{\mathrm{inv}}$ of degree 1 such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{T}_{\mathrm{dist}} \arrow[dl] \arrow[d] & \\ (S^+\mathfrak{g}^*)_{\mathrm{inv}} & \mathcal{P}^* \arrow[l, "\gamma"] & \subset (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \end{tikzcd}$$

The map γ identifies $\tilde{\mathcal{P}}^*$ as a subspace of $(S\mathfrak{g}^*)_{\mathrm{inv}}$. By Chevalley's theorem, the inclusion map extends to an algebra isomorphism, $S\tilde{\mathcal{P}}^* \cong (S\mathfrak{g}^*)_{\mathrm{inv}}$.

Below we will also need another description of the space $\mathcal{P}^*$ of primitive elements. Let

$$\varsigma\colon S\mathfrak{g}^* \to \wedge\mathfrak{g}^* \tag{14}$$

be the homomorphism of graded algebras, given on $S^1\mathfrak{g}^* = \mathfrak{g}^*$ by the Lie algebra differential. View $\wedge\mathfrak{g}^*$ as a module for the subalgebra $\mathrm{im}(\varsigma)$, and let $\mathrm{im}(\varsigma)\mathfrak{g}^*$ be the submodule generated by $\mathfrak{g}^*$.

Lemma 2.4. *The space of primitive elements in* $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ *is the invariant subspace of* $\mathrm{im}(\varsigma)\mathfrak{g}^*$*:*

$$\mathcal{P}^* = (\mathrm{im}(\varsigma)\mathfrak{g}^*)_{\mathrm{inv}}.$$

Proof. It is well-known (cf. [7, page 233] or [13, Equation (261)]) that for any polynomial $p \in (S^i\mathfrak{g}^*)_{\mathrm{inv}}$ the element $\tau(p)$ is given, up to a multiplicative constant, by

$$\sum_a \varsigma(\iota^S(e_a)p) \wedge e^a \in (\wedge^{2i-1}\mathfrak{g}^*)_{\mathrm{inv}}.$$

Here $\iota^S(\xi)$ is the derivation of $S\mathfrak{g}^*$, given on $S^1\mathfrak{g}^* = \mathfrak{g}^*$ by the natural pairing. (Put differently, $\iota^S(\xi)p$ is the derivative of the polynomial p in the direction of ξ.) This gives the inclusion $\mathcal{P}^* \subset (\mathrm{im}(\varsigma)\mathfrak{g}^*)_{\mathrm{inv}}$. Since $\mathfrak{g}$ is reductive, the map

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{im}(\varsigma)) \cong (\mathrm{im}(\varsigma) \otimes \mathfrak{g}^*)_{\mathrm{inv}} \to (\mathrm{im}(\varsigma)\mathfrak{g}^*)_{\mathrm{inv}}$$

given by wedge product is onto. According to Kostant [13, Equation (263)], the multiplicity of the adjoint representation in $\mathrm{im}(\varsigma)$ equals $\mathrm{rank}(\mathfrak{g}) = \dim \mathcal{P}^*$. We conclude

$$\dim(\mathrm{im}(\varsigma)\mathfrak{g}^*)_{\mathrm{inv}} \le \dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{im}(\varsigma)) = \dim \mathcal{P}^*.$$

□

3. A canonical cochain of transgression

3.1. **The inclusion $K(\mathcal{P}) \hookrightarrow W\mathfrak{g}$.** The action of $\wedge\mathfrak{g}$ on $\wedge\mathfrak{g}^*$ by contractions ι extends to an action (still denoted ι) of the graded algebra $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}$ on the Weil algebra $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$. Lemma 2.1 shows that on the invariant subspace $(W\mathfrak{g})_{\mathrm{inv}}$, the action of $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$ commutes with the differential. That is, $(W\mathfrak{g})_{\mathrm{inv}}$ is a differential graded module over the algebra $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$.

Let $c^j \in \mathcal{P}^*$ and $c_j \in \mathcal{P}$ denote dual (homogeneous) bases for the primitive subspaces. Let $p^j = \gamma(c^j) \in \tilde{\mathcal{P}}^*$ denote the corresponding generators of $(S\mathfrak{g}^*)_{\mathrm{inv}}$. The Koszul algebra of $\mathcal{P}$ is the tensor product

$$K(\mathcal{P}) = S\tilde{\mathcal{P}}^* \otimes \wedge\mathcal{P}^*,$$

with differential $\mathrm{d}^K = \sum_j p^j \otimes \iota(c_j)$. Thus d^K vanishes on $S\tilde{\mathcal{P}}^*$, and takes the generators c^j of $\wedge\mathcal{P}^*$ to the corresponding generators p^j of $S\tilde{\mathcal{P}}^*$. The Koszul algebra is a differential graded modules for the algebra

$$S\tilde{\mathcal{P}}^* \otimes \wedge\mathcal{P},$$

where the first factor acts by multiplication and the second factor acts by contraction. The natural inclusion $K(P) \hookrightarrow (W\mathfrak{g})_{\mathrm{inv}}$ is compatible with the module structures, but is not a cochain map. However, we have the following result:

Theorem 3.1. *There is an injective homomorphism of graded vector spaces*

$$\Phi\colon K(\mathcal{P}) \to (W\mathfrak{g})_{\mathrm{inv}},$$

with the following properties:

(a) Φ *is a cochain map,*
(b) Φ *is a homomorphism of modules over* $S\tilde{\mathcal{P}}^* \otimes \wedge\mathcal{P} \cong (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$*,*
(c) Φ *is unital, that is, it takes the unit of* $K(P)$ *to the unit of* $(W\mathfrak{g})_{\mathrm{inv}}$*.*

As a direct consequence of Theorem 3.1, we have:

Corollary 3.2. *The restriction of the map* Φ *to* $\mathcal{P}^* = \wedge^1\mathcal{P}^* \subset K(P)$ *fits into a commutative diagram,*

$$\begin{tikzcd} & \mathcal{T}_{\mathrm{dist}} \arrow[dl] & \\ (S^+\mathfrak{g}^*)_{\mathrm{inv}} & \mathcal{P}^* \arrow[l, "\gamma"] \arrow[u, "\Phi"'] & \subset (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \end{tikzcd}$$

Proof. The element $\Phi(c^j) \in (W\mathfrak{g})_{\mathrm{inv}}$ satisfies

$$\mathrm{d}^W \Phi(c^j) = \Phi(\mathrm{d}^K c^j) = \Phi(p^j) = p^j\Phi(1) = p^j = \gamma(c^j).$$

Since furthermore $\iota(c_i)\Phi(c^j) = \Phi(\iota(c_i)c^j) = \delta_i^j$, it follows that $\Phi(c^j)$ is a distinguished cochain of transgression. $\square$

Remark 3.3. On the other hand, let $\tilde{c}^j \in (W\mathfrak{g})_{\mathrm{inv}}$ be distinguished cochains of transgression extending c^j, and consider the algebra homomorphism,

$$\Phi'\colon K(P) \to (W\mathfrak{g})_{\mathrm{inv}}, \quad \Phi'(p^j) = p^j,\ \Phi'(c^j) = \tilde{c}^j.$$

Then Φ' is a homomorphism of differential $S\tilde{P}^*$-algebras, but it does not intertwine the $\wedge P$-actions (unless $\mathfrak{g}$ is Abelian). Indeed, recall that the operators $\iota(c_j)$ are derivations of $K(P)$ but not of $(W\mathfrak{g})_{\mathrm{inv}}$.

3.2. **The map Φ.** In this Section we determine the most general form of a map Φ which satisfies conditions (b) and (c), and reduce the condition (a) to a Maurer-Cartan type equation. Note that for any even element $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$, the map

$$\Phi\colon S\tilde{\mathcal{P}}^* \otimes \wedge\mathcal{P}^* \cong (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \hookrightarrow (W\mathfrak{g})_{\mathrm{inv}} \xrightarrow{e^{\iota(f)}} (W\mathfrak{g})_{\mathrm{inv}} \tag{15}$$

satisfies properties (b) and (c). Furthermore, Φ preserves degrees if and only if f has degree 0. The following converse was pointed out to us by M. Franz:

Lemma 3.4. *Any even linear map $\Phi\colon K(P)\to(W\mathfrak{g})_{\mathrm{inv}}$ satisfying conditions* (b) *and* (c) *is of the form* (15)*, for a unique even element $f\in(S\mathfrak{g}^*\otimes(\wedge\mathfrak{g})^-)_{\mathrm{inv}}$. Moreover, Φ preserves degrees if and only if f has degree* 0*.*

Proof. Observe that any element of $K(P)$ is obtained from the volume element $c^1\cdots c^l$ by the action of $S\tilde{\mathcal{P}}^*\otimes\wedge\mathcal{P}$. Hence, a map Φ satisfying (b) is uniquely determined by $\Phi(c^1\cdots c^l)$. Similarly, any element in $W\mathfrak{g}$ is obtained from $c^1\cdots c^l$ (now viewed as a volume element in $\wedge\mathfrak{g}^*$) by the action of $S\mathfrak{g}^*\otimes\wedge\mathfrak{g}$. Let $F\in(S\mathfrak{g}^*\otimes\wedge\mathfrak{g})_{\mathrm{inv}}$ be the unique element such that

$$\iota(F)(c^1\cdots c^l)=\Phi(c^1\cdots c^l).$$

Then Φ is a composition of the inclusion map $K(P)\hookrightarrow(W\mathfrak{g})_{\mathrm{inv}}$ with $\iota(F)$. The image of the unit $1\in K(P)$ under this map equals the component of F in $(S\mathfrak{g}^*\otimes\wedge^0\mathfrak{g})_{\mathrm{inv}}\cong(S\mathfrak{g}^*)_{\mathrm{inv}}$. Hence, by (c) this component must be equal to 1. Equivalently, $F=e^f$ where $f\in(S\mathfrak{g}^*\otimes(\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ is given by

$$f=\log(F)=-\sum_n(1-F)^n/n.$$

(This is well-defined since $1-F$ is nilpotent.) If Φ preserves the grading, F as defined above has degree 0, hence also $f=\log(F)$ has degree 0. $\square$

Our next task is to arrange that Φ is a cochain map.

Proposition 3.5. *For any even element $f\in(S\mathfrak{g}^*\otimes(\wedge\mathfrak{g})^-)_{\mathrm{inv}}$, the conjugate of the Weil differential by $e^{-\iota(f)}$ is given by the formula,*

$$\mathrm{Ad}(e^{-\iota(f)})\mathrm{d}^W=\mathrm{d}^W+\iota\big(\partial f+\tfrac12[f,f]_{\wedge\mathfrak{g}}\big)+\sum_a\iota\big(\iota^*(e^a)f\big)\,L^S(e_a). \tag{16}$$

Proof. The first two terms in Formula (13) for the Weil differential contribute

$$\begin{aligned}\mathrm{Ad}\left(e^{-\iota(f)}\right)\Big(\sum_a y^aL^W(e_a)\Big)&=\sum_a y^aL^W(e_a)+\sum_a\iota\big(\iota^*(e^a)f\big)L^W(e_a),\\ \mathrm{Ad}\left(e^{-\iota(f)}\right)(-\mathrm{d}^\wedge)&=-\mathrm{d}^\wedge+\iota(\partial f+\tfrac12[f,f]_{\wedge\mathfrak{g}})-\sum_a\iota(\iota^*(e^a)f)L^\wedge(e_a)\end{aligned}$$

(using $\mathrm{Ad}(e^{-\iota(f)})y^a=y^a+\iota(\iota^*(e^a)f)$ and (10)), while the last term $\sum_a v^a\iota(e_a)$ in (13) commutes with the action of $e^{-\iota(f)}$. Equation (16) follows. $\square$

By (16) and the formula (13) for the Weil differential, we obtain

$$\mathrm{Ad}(e^{-\iota(f)})\mathrm{d}^W=\sum_a v^a\otimes\iota(e_a)+\iota\big(\partial f+\tfrac12[f,f]_{\wedge\mathfrak{g}}\big)+\cdots,$$

where the dots indicate terms vanishing on $(S\mathfrak{g}^*)_{\mathrm{inv}}\otimes(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. Hence $\Phi=e^{\iota(f)}$ will give the desired cochain map $K(\mathcal{P})\to(W\mathfrak{g})_{\mathrm{inv}}$, provided $f\in(S\mathfrak{g}^*\otimes(\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ has degree 0 and solves the inhomogeneous Maurer-Cartan equation,

$$\partial f+\tfrac12[f,f]_{\wedge\mathfrak{g}}=\sum_j p^j\otimes c_j-\sum_a v^a\otimes e_a. \tag{17}$$

Theorem 3.6. *The inhomogeneous Maurer-Cartan equation* (17) *has a (canonical) solution* $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ *of degree* 0*. In fact,* $Z = \sum_j p^j \otimes c_j$ *is the* only *element in* $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$ *for which the equation*

$$\partial f + \tfrac{1}{2}[f,f]_{\wedge\mathfrak{g}} + \sum_a v^a \otimes e_a = Z$$

admits such a solution.

Corollary 3.2 may now be restated as the assertion that for any even, nilpotent solution f of Equation (17) and any $c^j \in \mathcal{P}^*$, the element $\tilde{c}^j = e^{\iota(f)} c^j$ is a distinguished cochain of transgression, with $\mathrm{d}^W \tilde{c}^j = p^j$.

The proof of Theorem 3.6 will be given at the end of Section 3.4, after some preparations. As a consequence of Theorem 3.6, we recover Chevalley's correspondence between primitive generators of $(\wedge\mathfrak{g})_{\mathrm{inv}}$ and generators of $(S\mathfrak{g}^*)_{\mathrm{inv}}$ from a rather unexpected angle: It takes the form of a solvability condition for an inhomogeneous Maurer-Cartan equation in a differential graded Lie algebra.

Let us introduce the following notation,

$$\begin{aligned} \mathfrak{k} &= \bigoplus_{i\le 0} \mathfrak{k}^i, \quad \mathfrak{k}^i = (S\mathfrak{g}^* \otimes \wedge^{1-i}\mathfrak{g})_{\mathrm{inv}}, \\ \mathfrak{l} &= \bigoplus_{i\le 0} \mathfrak{l}^i, \quad \mathfrak{l}^i = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge^{1-i}\mathfrak{g})_{\mathrm{inv}}, \\ X &= -\sum_a v^a \otimes e_a. \end{aligned} \tag{18}$$

As a first step toward solving (17), we will look for solutions of $\partial f + \frac{1}{2}[f,f]_{\mathfrak{k}} = X$ *modulo* $\mathfrak{l}$. This will be done in Section 3.3 below. We will need the following fact:

Lemma 3.7. *The element* $X = -\sum_a v^a \otimes e_a$ *is a cocycle, contained in the center* $\mathfrak{z}$*.*

Proof. It is clear that X is a cocycle, since ∂ vanishes on $\mathfrak{g} \subset \wedge\mathfrak{g}$. Furthermore, using invariance of elements $\phi \in \mathfrak{k}$ under the diagonal action,

$$[X,\phi]_{\wedge\mathfrak{g}} = -\sum_a v^a L^\wedge(e_a)\phi = \sum_a v^a L^S(e_a)\phi = 0.$$

Here we used the fact that the derivation $\sum_a v^a L^S(e_a)$ of $S\mathfrak{g}^*$ is zero. □

Observe that $\mathfrak{l}$ is contained in the center $\mathfrak{z}$ of $\mathfrak{k}$, and that ∂ vanishes on $\mathfrak{l}$. Furthermore, since $(\wedge\mathfrak{g})_{\mathrm{inv}} \hookrightarrow \wedge\mathfrak{g}$ is a $\mathfrak{g}$-equivariant homotopy equivalence, the inclusion of $\mathfrak{l}$ into $\mathfrak{k}$ is a homotopy equivalence.

3.3. **Solutions of a Maurer-Cartan equation.** It is convenient to place (17) into a more general framework. Let $\mathfrak{k} = \bigoplus_{i\le 0} \mathfrak{k}^i$ be a differential graded Lie algebra with a differential ∂ of degree $+1$. We assume that the grading is bounded below, that is $\mathfrak{k}^i = 0$ for $i << 0$. Denote by $U(\mathfrak{k}) = \bigoplus_{i\le 0} U(\mathfrak{k})^i$ the enveloping algebra, with grading induced by the grading of $\mathfrak{k}$ (that is, $\deg(x_1 \cdots x_r) = i_1 + \ldots + i_r$ for $x_j \in \mathfrak{k}^{i_j}$), and by $\overline{U}(\mathfrak{k}) = \prod_{i\le 0} U(\mathfrak{k})^i$ the degree completion of $U(\mathfrak{k})$. Elements of $\overline{U}(\mathfrak{k})$ are infinite series $a = \sum_{i\le 0} a_i$ with $a_i \in U(\mathfrak{k})^i$. The differential ∂ extends to $\overline{U}(\mathfrak{k})$ as a derivation of the product. Write $\mathfrak{k}^- = \bigoplus_{i<0} \mathfrak{k}^i$. Its even

part $\mathfrak{k}^-_{\mathrm{even}}$ is an (ordinary) nilpotent Lie algebra.[2] There is a well-defined exponential map $\exp: \mathfrak{k}^-_{\mathrm{even}} \to \overline{U}(\mathfrak{k}), s \mapsto \exp(s) = \sum_N \frac{1}{N!} s^N$. It is a 1-1 map, and its image in $\overline{U}(\mathfrak{k})$ is a group, with product given by the Campbell-Hausdorff formula. One has the following well-known formula for the (right) *Maurer-Cartan form*,

$$(\partial \exp(s)) \exp(-s) = j^R(\mathrm{ad}_s)\partial s, \quad s \in \mathfrak{k}^-_{\mathrm{even}}, \tag{19}$$

where $j^R(z) = \frac{e^z - 1}{z}$ and $\mathrm{ad}_s = [s, \cdot]_{\mathfrak{k}}$. The Maurer-Cartan form enters into the formula for the *gauge action* of $\exp(\mathfrak{k}^-_{\mathrm{even}})$ on $\mathfrak{k}^-_{\mathrm{odd}}$,

$$\exp(s).f = e^{\mathrm{ad}_s} f - j^R(\mathrm{ad}_s)\partial s. \tag{20}$$

The *curvature* $\partial f + \frac{1}{2}[f, f]_{\mathfrak{k}}$ of any $f \in \mathfrak{k}^-_{\mathrm{odd}}$ transforms under the adjoint representation: If $\tilde{f} = \exp(s).f$, then

$$\partial \tilde{f} + \tfrac{1}{2}[\tilde{f}, \tilde{f}]_{\mathfrak{k}} = e^{\mathrm{ad}_s}\big(\partial f + \tfrac{1}{2}[f, f]_{\mathfrak{k}}\big). \tag{21}$$

Theorem 3.8. *Let $(\mathfrak{k}, \partial)$ be a differential graded Lie algebra, with $\mathfrak{k}^i = 0$ for $i << 0$ and $\mathfrak{k}^i = 0$ for $i > 0$. Assume there is a subspace $\mathfrak{l}$ of the center $\mathfrak{z}$ of $\mathfrak{k}$, such that ∂ vanishes on $\mathfrak{l}$ and such that the inclusion $\mathfrak{l} \hookrightarrow \mathfrak{k}$ induces an isomorphism in cohomology. Then:*

(a) *For any even central element $X \in \mathfrak{z}$ with $\partial X = 0$, the set of solutions $f \in \mathfrak{k}^-_{\mathrm{odd}}$ of the equation*

$$\partial f + \tfrac{1}{2}[f, f]_{\mathfrak{k}} = X \mod \mathfrak{l} \tag{22}$$

is a homogeneous space for the group $\exp(\mathfrak{k}^-_{\mathrm{even}}) \times \mathfrak{l}^-_{\mathrm{odd}}$*, where the first factor acts by gauge transformations and the second factor by translations.*

(b) *The difference*

$$\partial f + \tfrac{1}{2}[f, f]_{\mathfrak{k}} - X \in \mathfrak{l}$$

is independent of the solution f.

Proof. For any solution f of (22) the curvature $\partial f + \frac{1}{2}[f, f]_{\mathfrak{k}}$ is in the center of $\mathfrak{k}$. It is therefore invariant under gauge transformations of f (see (21)). On the other hand, the curvature is also invariant under the translation action of $\mathfrak{l}^-_{\mathrm{odd}}$ (since element of $\mathfrak{l}$ are central cocycles by assumption). Hence (b) follows from (a).

To prove (a) we have to show that the solution space is non-empty and that the action of $\exp(\mathfrak{k}^-_{\mathrm{even}}) \times \mathfrak{l}^-_{\mathrm{odd}}$ is transitive. It suffices to prove these statements for the quotient $\mathfrak{k}/\mathfrak{l}$. Equivalently, we may (and will) assume for the rest of this proof that $\mathfrak{l} = 0$, hence $H(\mathfrak{k}) = 0$. Write $f = f_1 + f_3 + \ldots$ with $f_i \in \mathfrak{k}^{-i}$, and similarly $X = X_0 + X_2 + \ldots$ with $X_i \in \mathfrak{k}^{-i}$. Then (22) is equivalent to a system of equations,

$$\partial f_N = -\tfrac{1}{2} \sum_{i+j=N-1} [f_i, f_j]_{\mathfrak{k}} + X_{N-1}, \tag{A_N}$$

where $N = 1, 3, \ldots$. Equation (A_1) reads $\partial f_1 = X_0$. It admits a solution since $\partial X_0 = 0$ and since $H(\mathfrak{k}) = 0$. Suppose by induction that we have found solutions f_i for the system of

[2]In the previous Section, the labels 'even' and 'odd' referred to the grading on $\wedge\mathfrak{g}$. Since (18) involves the shifted grading $\wedge\mathfrak{g}[1]$, the roles of 'even' and 'odd' are now reversed.

equations (A_i) up to $i = N-2$, and consider (A_N). We have,

$$\partial\big(-\tfrac{1}{2}\sum_{i+j=N-1}[f_i,f_j]_{\mathfrak{k}} + X_{N-1}\big) = -\sum_{i+j=N-1}[\partial f_i, f_j]_{\mathfrak{k}} = -\sum_{i+j=N-1}[X_{i-1}, f_j]_{\mathfrak{k}} = 0.$$

Here we have used the Jacobi identity for $\mathfrak{k}$, and the assumption that X is central. Since $H(\mathfrak{k}) = 0$, it follows that (A_N) admits a solution. This proves the existence part of the theorem. To show uniqueness up to gauge transformations, suppose $f \in \mathfrak{k}^-_{\mathrm{odd}}$ is a solution of (22). Then the derivation $\nabla = \partial + \mathrm{ad}_f$ is again a differential on $\mathfrak{k}$:

$$\nabla^2 = \mathrm{ad}(\partial f + \tfrac{1}{2}[f,f]_{\mathfrak{k}}) = \mathrm{ad}(X) = 0.$$

Given $r \in \mathfrak{k}^-_{\mathrm{odd}}$, the sum $f + r$ is a solution of (22) if and only if

$$\nabla r + \tfrac{1}{2}[r,r]_{\mathfrak{k}} = 0. \tag{23}$$

We must show that $f + r$ is gauge equivalent to f. The formula (20) for gauge transformations of f can be written

$$\exp(s).f = f - j^R(\mathrm{ad}_s)\nabla s, \tag{24}$$

where we used (19) and the identity $e^{\mathrm{ad}_s} f = f + j^R(\mathrm{ad}_s)[s,f]_{\mathfrak{k}}$. Hence, we have to prove that

$$r = -j^R(\mathrm{ad}_s)\nabla s \tag{25}$$

for some $s = s_2 + s_4 + \cdots$ with $s_i \in \mathfrak{k}^{-i}$. Write

$$s_{\{N\}} = s_2 + s_4 + \cdots + s_N, \quad r_{\{N\}} = -j^R(\mathrm{ad}_{s_{\{N\}}})\nabla s_{\{N\}}.$$

Suppose we have found $s_2, s_4, \ldots, s_N$, such that $q_{\{N\}} = r - r_{\{N\}}$ is contained in $\mathfrak{k}^{-(N+1)} \oplus \mathfrak{k}^{-(N+3)} + \cdots$. Since $r_{\{N\}}$ solves the Maurer-Cartan equation (23),

$$\nabla q_{\{N\}} + \tfrac{1}{2}[q_{\{N\}}, q_{\{N\}}]_{\mathfrak{k}} + [r_{\{N\}}, q_{\{N\}}]_{\mathfrak{k}} = 0.$$

By construction, the left hand side lies in $\mathfrak{k}^{-N} \oplus \mathfrak{k}^{-(N+2)} + \cdots$. Moreover, the only contribution to the component in $\mathfrak{k}^{-N}$ comes from $\partial q_{\{N\}}$. It follows that the component of $q_{\{N\}}$ in $\mathfrak{k}^{-(N+1)}$ is closed, hence exact (since $H(\mathfrak{k}) = 0$). Choose $s_{N+2} \in \mathfrak{k}^{-(N+2)}$ such that the component of $q_{\{N\}} - \partial s_{N+2}$ in $\mathfrak{k}^{-(N+1)}$ is zero. Letting $s_{\{N+2\}} := s_{\{N\}} + s_{N+2}$, we achieve $q_{\{N+2\}} \in \mathfrak{k}^{-(N+3)} \oplus \mathfrak{k}^{-(N+5)} + \cdots$. Hence, by induction we obtain the desired element s. $\square$

Assume that $\mathcal{S}\colon \mathfrak{k} \to \mathfrak{k}$ is a homotopy operator, i.e. $\mathcal{S}$ has degree -1 and $[\mathcal{S}, \partial] = I - \Pi$ where Π is a projection operator onto $\mathfrak{l}$. Then we can write down an explicit solution to (22), by the following recursion formula:

$$f_N = \mathcal{S}\big(-\tfrac{1}{2}\sum_{i+j=N-1}[f_i,f_j]_{\mathfrak{k}} + X_{N-1}\big), \quad N = 1, 3, \ldots \tag{26}$$

3.4. **Solution of Equation** (17)**.** We now return to our original problem, Equation (17). Choose an invariant scalar product B on $\mathfrak{g}$, and let $\mathcal{S} = \delta\mathcal{G}$ be the homotopy operator defined by Hodge theory (Section 2.3). Let $f = f_1 + f_3 + \ldots$ be the solution *modulo* $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$, given by the recursion formula (26).

Lemma 3.9. *The element $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ has total degree* 0*.*

Here *total degree* refers to our original grading on the algebra $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}$, in contrast to the Lie algebra grading used in (18).

Proof. The element $X = \sum_a v^a \otimes e_a$ has total degree 1, while the homotopy operator $\mathcal{S}$ has total degree -1. Thus $f_1 = \mathcal{S}(X)$ has total degree 0. Since the Schouten bracket of any two elements of total degree 0 has total degree -1, the recursion formula (26) shows that all f_N have total degree 0. $\square$

To complete the proof of Theorem 3.6, it remains to identify the 'error term'

$$Z = \partial f + \tfrac{1}{2}[f, f]_{\wedge\mathfrak{g}} + \sum_a v^a \otimes e_a \in (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}. \tag{27}$$

Recall that by Theorem 3.8(b), Z is independent the choice of solution f.

Using the isomorphism $B^\flat\colon \mathfrak{g} \to \mathfrak{g}^*$, the map (14) translates into an algebra homomorphism,

$$\zeta\colon S\mathfrak{g} \to \wedge\mathfrak{g}.$$

extending the map $\delta\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$. View $\wedge\mathfrak{g}$ as a module for the subalgebra $\mathrm{im}(\zeta)$, and let $\mathrm{im}(\zeta)\mathfrak{g}$ denote the submodule generated by $\mathfrak{g} = \wedge^1\mathfrak{g}$. Notice that

$$\begin{aligned}
\iota^*(B^\flat(\xi))\colon\ & \mathrm{im}(\zeta) \to \mathrm{im}(\zeta)\mathfrak{g},\\
L(\xi)\colon\ & \mathrm{im}(\zeta) \to \mathrm{im}(\zeta),\\
\delta\colon\ & \mathrm{im}(\zeta)\mathfrak{g} \to \mathrm{im}(\zeta),\\
\partial\colon\ & \mathrm{im}(\zeta) \to \mathrm{im}(\zeta)\mathfrak{g}.
\end{aligned}$$

It follows that $\mathrm{im}(\zeta)$ and $\mathrm{im}(\zeta)\mathfrak{g}$ are both invariant under $\mathcal{L} = [\delta, \partial]$, and that the Schouten bracket of any two elements in $\mathrm{im}(\zeta)$ is contained in $\mathrm{im}(\zeta)\mathfrak{g}$.

Remark 3.10. The subspace $\mathrm{im}(\zeta)$ does not depend on the choice of the invariant scalar product B. If $\mathfrak{g}$ is simple, this follows since B is unique up to a multiplicative constant in this case, and δ just scales by that constant. For the general case, it suffices to observe that if $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with scalar product $B = \bigoplus B_i$, then the subalgebra $\mathrm{im}(\zeta)$ is generated by the images of the differentials δ_i, obtained from the Lie algebra differential on $\mathfrak{g}_i$ by the isomorphism $B_i^\flat\colon \mathfrak{g}_i \to \mathfrak{g}_i^*$.

Lemma 3.11. *The solution $f \in (S\mathfrak{g}^* \otimes \wedge\mathfrak{g})_{\mathrm{inv}}$ defined by the homotopy operator $\mathcal{S} = \delta\mathcal{G}$ is contained in $(S\mathfrak{g}^* \otimes \mathrm{im}(\zeta))_{\mathrm{inv}}$. Hence,*

$$Z \in (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\mathrm{im}(\zeta)\mathfrak{g})_{\mathrm{inv}} = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{P}.$$

Proof. We use the notation from Section 2.3. Since $\mathrm{im}(\zeta)$ is invariant under $\mathcal{L}$, it is also invariant under the Green's operator $\mathcal{G}$. Hence, $\mathcal{S}\colon (S\mathfrak{g}^* \otimes \mathrm{im}(\zeta)\mathfrak{g})_{\mathrm{inv}} \to (S\mathfrak{g}^* \otimes \mathrm{im}(\zeta))_{\mathrm{inv}}$. Obviously, $X = -\sum_a v^a \otimes e_a \in (S\mathfrak{g}^* \otimes \mathrm{im}(\zeta)\mathfrak{g})_{\mathrm{inv}}$. An induction based on the recursive definition (26) therefore shows that each f_n is in $(S\mathfrak{g}^* \otimes \mathrm{im}(\zeta))_{\mathrm{inv}}$. This proves the first claim. The properties of ζ show $Z \in S\mathfrak{g}^* \otimes \mathrm{im}(\zeta)\mathfrak{g}$, while on the other hand $Z \in (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$. Finally, $(\mathrm{im}(\zeta)\mathfrak{g})_{\mathrm{inv}} = \mathcal{P}$ by Lemma 2.4. $\square$

End of Proof of Theorem 3.6. Using the lemma we may write $Z = \sum_j q^j \otimes c_j$, where the $c_j \in (\wedge\mathfrak{g})_{\mathrm{inv}}$ are a homogeneous basis of $\mathcal{P}$, while the $q^j \in (S\mathfrak{g}^*)_{\mathrm{inv}}$ are invariant polynomials. To determine the elements q^j, let us once again consider $e^{\iota(f)}$ as an operator on $(W\mathfrak{g})_{\mathrm{inv}}$. By Proposition 3.5 we have

$$\mathrm{Ad}(e^{-\iota(f)})(\mathrm{d}^W) = \sum_j q^j \otimes \iota(c_j) + \ldots = \iota(Z) + \ldots$$

where $\ldots$ are terms vanishing on $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. Apply this result to $c^k \in \mathcal{P}^* = (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. The element $\tilde{c}^k = e^{\iota(f)} c^k \in (W\mathfrak{g})_{\mathrm{inv}}$ satisfies,

$$\begin{aligned}
\mathrm{d}^W \tilde{c}^k &= \mathrm{d}^W e^{\iota(f)} c^k = e^{\iota(f)} \iota(Z) c^k = e^{\iota(f)} q^k = q^k, \\
\iota(c_j)\tilde{c}^k &= \iota(c_j) e^{\iota(f)} c^k = e^{\iota(f)} \delta_j^k = \delta_j^k.
\end{aligned}$$

Thus, the $\tilde{c}^k$ are distinguished cochains of transgression. In particular, $q^k = \mathrm{d}^W \tilde{c}^k = \gamma(c^k) = p^k$. This concludes the proof of Theorem 3.6. □

According to Theorem 3.8, the solution f of the Maurer-Cartan equation (17) is unique up to gauge transformation and translation by elements in $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})_{\mathrm{inv}}$. The special solution found above is singled out by the 'gauge fixing'. More precisely:

Proposition 3.12. *The Maurer-Cartan equation* (17) *admits a unique δ-exact solution $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$, given explicitly by the recursion formula* (26) *with $\mathcal{S} = \delta\mathcal{G}$. This solution does not depend on the choice of invariant scalar product B (even though δ does).*

Proof. Recall that the Maurer-Cartan equation (17) is equivalent to a system of equations of the form (A_N) for $f = f_1 + f_3 + \cdots \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$,

$$\partial f_N = -\tfrac{1}{2} \sum_{i+j=N-1} [f_i, f_j]_{\mathfrak{k}} + \Big(\sum_j p^j \otimes c_j - \sum_a v^a \otimes e_a \Big)_{N-1}.$$

At each stage, the recursion (26) picks out the unique (by the Hodge decomposition (12)) δ-exact solution. As shown in Lemma 3.11, this solution f is in fact contained in the subspace $(S\mathfrak{g}^* \otimes \mathrm{im}(\zeta))_{\mathrm{inv}} \subset \mathrm{im}(\delta)_{\mathrm{inv}}$. By Remark 3.10 this subspace does not depend on B. Hence f does not depend on B. □

The first term f_1 in our recursion formula for f can be computed quite easily. Suppose that $\mathfrak{g}$ is simple. On $\mathfrak{g} \subset \wedge\mathfrak{g}$, the operator $\mathrm{Cas}^{\wedge}_{\mathfrak{g}}$ acts as a scalar, $(\dim\mathfrak{g})^{-1}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}})$. Hence

$$f_1 = -\delta\mathcal{G}(\sum_a v^a e_a) = \frac{2\dim\mathfrak{g}}{\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}})} \sum_a v^a \delta e_a.$$

Example 3.13. Let $\mathfrak{g}$ be the three-dimensional Lie algebra with basis e_1, e_2, e_3 and bracket relations, $[e_1, e_2]_{\mathfrak{g}} = e_3$, $[e_2, e_3]_{\mathfrak{g}} = e_1$, $[e_3, e_1]_{\mathfrak{g}} = e_2$. Using the inner product B on $\mathfrak{g}$ for which the e_a are an orthonormal basis, we find $\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}}) = -6$. Hence,

$$f_1 = -\delta(\sum_a v^a e_a) = v^1 \otimes (e_2 \wedge e_3) + v^2 \otimes (e_3 \wedge e_1) + v^3 \otimes (e_1 \wedge e_2).$$

In this example, higher corrections do not appear, so that $f = f_1$. Indeed, taking $c = e_1 \wedge e_2 \wedge e_3$ as a generator of $(\wedge\mathfrak{g})_{\mathrm{inv}}$, a short calculation shows

$$\tfrac{1}{2}[f,f]_{\wedge\mathfrak{g}} = p \otimes c, \quad \partial f = -\sum_a v^a e_a,$$

where $p = \sum_a v^a v^a \in (S\mathfrak{g}^*)_{\mathrm{inv}}$.

In all of our applications, the solution f of the Maurer-Cartan equation enters via its exponential in the algebra $(S\mathfrak{g}^* \otimes \wedge\mathfrak{g})_{\mathrm{inv}}$. We will now give an explicit formula for the exponential.

Proposition 3.14. *The exponential of the solution f is given by the formula,*

$$e^f = (I + \mathcal{G} \circ \sum_a v^a \delta e_a)^{-1}(1).$$

The inverse is well-defined, since $\mathcal{G} \circ \sum_a v^a \delta e_a$ is nilpotent.

Proof. Let $F = e^f$, and denote by $F_{[k]}$, $k = 0, 2, \ldots$ the component in $(S\mathfrak{g}^* \otimes \wedge^k\mathfrak{g})_{\mathrm{inv}}$. Since f is δ-exact, F is δ-closed, and each $F_{[k]}$ with $k \geq 2$ is δ-exact. Using Lemma 2.3, Equation (17) is equivalent to

$$F_{[0]} = 1, \quad \partial F = YF, \tag{28}$$

where $Y = -\sum_a v^a \otimes e_a + \sum_j p^j \otimes c_j$. Applying δ, we obtain $\mathcal{L}F = \delta\partial F = (\delta Y)F$. That is,

$$F_{[0]} = 1, \quad \mathcal{L}F_{[k+2]} = (\delta Y)F_{[k]}$$

because the Hodge Laplacian $\mathcal{L}$ preserves the exterior algebra degree, while $\delta Y = -\sum_a v^a \delta e_a \in (S\mathfrak{g}^* \otimes \wedge^2\mathfrak{g})_{\mathrm{inv}}$. Since $F_{[k+2]}$ is δ-exact, $F_{[k+2]} = \mathcal{G}\mathcal{L}F_{[k+2]}$. We arrive at the recursion formula

$$F_{[0]} = 1, \quad F_{[k+2]} = \mathcal{G}(\delta Y F_{[k]}),$$

with solution $F_{[k]} = (\mathcal{G} \circ \delta Y)^k(1)$. Summing $F = \sum_k F_{[k]}$ as a geometric series, the proof is complete. $\square$

4. The small Cartan complex

4.1. **$\mathfrak{g}$-differential spaces.** A $\mathfrak{g}$-*differential space* is a differential graded vector space $(\mathcal{M}, \mathrm{d})$, together with linear maps

$$L\colon \mathfrak{g} \to \mathrm{End}(\mathcal{M}), \quad \iota\colon \mathfrak{g} \to \mathrm{End}(\mathcal{M}) \tag{29}$$

such that the *Lie derivatives* $L(\xi)$ have degree 0 and the *contractions* $\iota(\xi)$ have degree -1, and such that the following relations hold:

$$\begin{aligned} [\mathrm{d}, \iota(\xi)] &= L(\xi), \\ [L(\xi), \iota(\xi')] &= \iota([\xi,\xi']_{\mathfrak{g}}), \\ [\iota(\xi), \iota(\xi')] &= 0. \end{aligned} \tag{30}$$

These relations and the Jacobi identity imply that the Lie derivatives $L(\xi)$ define a representation of $\mathfrak{g}$ on $\mathcal{M}$, commuting with the differential. The motivating example of a $\mathfrak{g}$-differential space is the space $\mathcal{M} = \Omega(M)$ of differential forms on a manifold M with an action of a Lie group G, with (29) the Lie derivatives and contractions by the infinitesimal generators of the

action. Another example is $\mathcal{M} = \wedge\mathfrak{g}^*$, with d the Lie algebra differential, $\iota(\xi)$ the usual contraction operators, and $L(\xi)$ the coadjoint representation. The Weil algebra $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ is a $\mathfrak{g}$-differential space, with d the Weil differential d^W, contractions $\iota(\xi) = 1 \otimes \iota(\xi)$, and Lie derivatives $L(\xi) = L^W(\xi)$.

For any $\mathfrak{g}$-differential space $\mathcal{M}$, one defines the horizontal subspace $\mathcal{M}_{\mathrm{hor}} = \bigcap_{\xi\in\mathfrak{g}} \ker(\iota(\xi))$, the invariant subspace $\mathcal{M}_{\mathrm{inv}} = \bigcap_{\xi\in\mathfrak{g}} \ker(L(\xi))$, and the basic subspace $\mathcal{M}_{\mathrm{basic}} = \mathcal{M}_{\mathrm{hor}} \cap \mathcal{M}_{\mathrm{inv}}$. Both $\mathcal{M}_{\mathrm{inv}}$ and $\mathcal{M}_{\mathrm{basic}}$ are stable under d.

The contraction operators on a $\mathfrak{g}$-differential space $\mathcal{M}$ extend to an algebra homomorphism $\iota\colon \wedge\,\mathfrak{g} \to \mathrm{End}(\mathcal{M})$. The formulas (9) and (10) hold in this greater generality, since their proof only relied on the commutation relations between the operators $L(\xi), \iota(\xi), \mathrm{d}$. Formula (9) shows that the action of $(\wedge\mathfrak{g})_{\mathrm{inv}}$ on $\mathcal{M}_{\mathrm{inv}}$ by contractions commutes with the differential. That is, $\mathcal{M}_{\mathrm{inv}}$ is a *differential* $(\wedge\mathfrak{g})_{\mathrm{inv}}$*-module.*

4.2. **The small Cartan model.** The *equivariant cohomology* $H_{\mathfrak{g}}(\mathcal{M})$ of any $\mathfrak{g}$-differential space $\mathcal{M}$ is defined as the cohomology of the Cartan complex

$$C_{\mathfrak{g}}(\mathcal{M}) = (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}},\ \mathrm{d}_{\mathfrak{g}} = 1 \otimes \mathrm{d} - \sum_a v^a \otimes \iota(e_a). \tag{31}$$

The algebra $(S\mathfrak{g}^*)_{\mathrm{inv}}$ of invariant polynomials acts on $C_{\mathfrak{g}}(\mathcal{M})$ by multiplication, and this action commutes with the differential. That is, $C_{\mathfrak{g}}(\mathcal{M})$ is a *differential* $(S\mathfrak{g}^*)_{\mathrm{inv}}$*-module.*

Definition 4.1. The *small Cartan model* for $\mathcal{M}$ is the differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-module

$$\tilde{C}_{\mathfrak{g}}(\mathcal{M}) = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{M}_{\mathrm{inv}},\quad \tilde{\mathrm{d}}_{\mathfrak{g}} = 1 \otimes \mathrm{d} - \sum_j p^j \otimes \iota(c_j). \tag{32}$$

Note that if the Lie algebra $\mathfrak{g}$ is Abelian, the Cartan model and the small Cartan model coincide. In general, we have:

Theorem 4.2. *Let $\mathcal{M}$ be any $\mathfrak{g}$-differential space. For any solution $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ of the Maurer-Cartan equation* (17)*, the composition*

$$\tilde{C}_{\mathfrak{g}}(\mathcal{M}) \hookrightarrow C_{\mathfrak{g}}(\mathcal{M}) \xrightarrow{e^{\iota(f)}} C_{\mathfrak{g}}(\mathcal{M}) \tag{33}$$

is a homotopy equivalence of differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules. In particular, it induces an isomorphism in cohomology, $\tilde{H}_{\mathfrak{g}}(\mathcal{M}) \to H_{\mathfrak{g}}(\mathcal{M})$.

Proof. By Equations (10) and (17), the operator $e^{\iota(f)}$ on $C_{\mathfrak{g}}(\mathcal{M})$ takes the Cartan differential $\mathrm{d}_{\mathfrak{g}}$ into

$$\begin{aligned} \mathrm{d}'_{\mathfrak{g}} &:= e^{-\iota(f)} \circ \Big(1 \otimes \mathrm{d} - \sum_a v^a \otimes \iota(e_a)\Big) \circ e^{\iota(f)} \\ &= 1 \otimes \mathrm{d} - p^j \otimes \iota(c_j) + \sum_a \iota(\iota^*(e^a)f) L^{\mathcal{M}}(e_a). \end{aligned} \tag{34}$$

On $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{M}_{\mathrm{inv}}$, the last term on the right hand side vanishes. This proves that $e^{\iota(f)}$ gives a cochain map $\tilde{C}_{\mathfrak{g}}(\mathcal{M}) \to C_{\mathfrak{g}}(\mathcal{M})$.

To construct a homotopy inverse, let $C = C_{\mathfrak{g}}(\mathcal{M})$ (with differential $\mathrm{d}'_{\mathfrak{g}}$) and $\tilde{C} = \tilde{C}_{\mathfrak{g}}(\mathcal{M})$ (with differential $\tilde{\mathrm{d}}_{\mathfrak{g}}$). We have to find a morphism of differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules $C \to \tilde{C}$ that

is homotopy inverse to the inclusion, by an $(S\mathfrak{g}^*)_{\mathrm{inv}}$-equivariant homotopy. Pick an invariant inner product B on $\mathfrak{g}$, and let $\mathrm{Cas}^S_{\mathfrak{g}}$ be the corresponding Casimir operator on $S\mathfrak{g}^*$ (i.e. the image of $\mathrm{Cas}_{\mathfrak{g}} \in U(\mathfrak{g})$ under the coadjoint representation). Then $S\mathfrak{g}^*$ splits as a direct sum of the kernel and image of $\mathrm{Cas}^S_{\mathfrak{g}}$, and the kernel is the space of invariants. Let $\mathcal{L}_0$ denote the restriction of $\mathrm{Cas}_{\mathfrak{g}} \otimes 1$ to invariants in $C = (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}}$. Then

$$C = \ker(\mathcal{L}_0) \oplus \mathrm{im}(\mathcal{L}_0)$$

where $\ker(\mathcal{L}_0) = \tilde{C}$. Let Π_0 be the projection from C onto $\ker(\mathcal{L}_0)$ along $\mathrm{im}(\mathcal{L}_0)$, and $\mathcal{G}_0$ the Green's operator, i.e. $\mathcal{G}_0\Pi_0 = 0$ and $\mathcal{L}_0\mathcal{G}_0 = \mathcal{G}_0\mathcal{L}_0 = 1 - \Pi_0$.

Introduce a filtration $0 = \mathcal{M}_{(0)} \subset \mathcal{M}_{(1)} \subset \cdots \subset \mathcal{M}_{(\dim \mathfrak{g}+1)} = \mathcal{M}$, where $\mathcal{M}_{(j)}$ is the subspace for which all contractions with elements in $\wedge^j \mathfrak{g}$ are zero. Since the filtration of $\mathcal{M}$ is $\mathfrak{g}$-invariant, it gives rise to a filtration $0 = C_{(0)} \subset C_{(1)} \subset \cdots \subset C_{(\dim \mathfrak{g}+1)} = C$. All terms in $\mathrm{d}'_{\mathfrak{g}}$, except for the first term $1 \otimes \mathrm{d}$, lower the filtration degree by at least 1. Consider the following operator on C,

$$h = -\sum_a L^S(e_a) \otimes \iota(B^\sharp(e^a))$$

and put $\mathcal{L} = [\mathrm{d}'_{\mathfrak{g}}, h]$. On C,

$$[1 \otimes \mathrm{d}, h] = -\sum_a L^S(e_a) \otimes L^{\mathcal{M}}(B^\sharp(e^a)) = \mathrm{Cas}_{\mathfrak{g}} \otimes 1 = \mathcal{L}_0.$$

Hence $\mathcal{L} = \mathcal{L}_0 + R$ where R lowers the filtration degree. From $h\Pi_0 = 0$ we deduce $\mathcal{L}\Pi_0 = 0$, and therefore

$$\mathcal{L} = \mathcal{L}(1 - \Pi_0) = (1 + R\mathcal{G}_0)\mathcal{L}_0.$$

But $1 + R\mathcal{G}_0$ is invertible since $R\mathcal{G}_0$ lowers the filtration degree. It follows that $C = \ker(\mathcal{L}) \oplus \mathrm{im}(\mathcal{L})$ with $\ker(\mathcal{L}) = \ker(\mathcal{L}_0) = \tilde{C}$. Let $\mathcal{G}$ denote the Green's operator for the cochain map $\mathcal{L}$, thus $\mathcal{G}\mathcal{L} = \mathcal{L}\mathcal{G} = 1 - \Pi$ where Π is the projection from C onto $\tilde{C}$ along $\mathrm{im}(\mathcal{L})$. Then $H := h\mathcal{G}$ satisfies $[\mathrm{d}'_{\mathfrak{g}}, H] = 1 - \Pi$. $\square$

Remarks 4.3. (a) The fact that the map (33) is a quasi-isomorphism also follows very quickly from the spectral sequences for the filtrations

$$F^j = \bigoplus_{i \geq j} (S^i\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}}, \quad \tilde{F}^j = \bigoplus_{i \geq j} (S^i\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{M}_{\mathrm{inv}},$$

provided the complex $\mathcal{M}$ is bounded below (to ensure convergence of the spectral sequence). Since (33) is filtration preserving, it induces a morphism of spectral sequences, $\phi_r \colon \tilde{E}_r \to E_r$. The 0th stage for both spectral sequences are the Cartan complexes themselves, with differential $1 \otimes \mathrm{d}$. Hence

$$\phi_1 \colon \tilde{E}_1 = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes H(\mathcal{M}) \to E_1 = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes H(\mathcal{M}).$$

Here we have used that

$$H(\mathcal{M}_{\mathrm{inv}}) = H(\mathcal{M})_{\mathrm{inv}} = H(\mathcal{M})$$

since $L(\xi) = [\iota(\xi), \mathrm{d}]$. The map ϕ_1 is just the identity map, since $e^{\iota(f)} - 1$ raises the filtration degree by at least two. Since ϕ_1 is an isomorphism, the map in cohomology $\tilde{H}_{\mathfrak{g}}(\mathcal{M}) \to H_{\mathfrak{g}}(\mathcal{M})$ is an isomorphism as well.

(b) The $\mathbb{Z}$-grading on $\mathcal{M}$ does not play any role in the proof – the results hold more generally for $\mathfrak{g}$-differential spaces that are only $\mathbb{Z}_2$-graded.
(c) The differential in the small Cartan model is not a derivation for the obvious product structure. However, it is possible to introduce a new (non-associative) product, such that the differential is a derivation and such that the induced product in cohomology is the standard one. This will be explored in Section 6. It was pointed out to us by M. Franz that by the homotopy equivalence between the two Cartan models, the algebra structure on the large Cartan model gives rise to an A_∞-structure on the small Cartan model. The relevant machinery is developed in a paper by Gugenheim and Lambe [8].

The small Cartan model is useful provided one has good control over the contraction operators $\iota(c_j)$. For instance, it may happen that these operators all vanish:

Example 4.4. Let $M = G/G^\sigma$ be a symmetric space, defined by an *inner* involutive automorphism σ of a connected reductive Lie group G. One example is the Grassmannian $\mathrm{Gr}_\mathbb{C}(k,n)$ of k-planes in $\mathbb{C}^n$, with $G = U(n)$, and σ the involution given as conjugation by a diagonal matrix with entries $(1,\ldots,1,-1,\ldots,-1)$ down the diagonal, with k plus signs and $n-k$ minus signs.

Recall that G-invariant forms on a symmetric space M are automatically closed, and hence that the cohomology ring of M is canonically isomorphic to the ring of invariant forms on M. Since σ is by assumption an inner automorphism, i.e. $\sigma = \mathrm{Ad}_x$ for some $x \in G$, the induced action on $(\wedge\mathfrak{g})_{\mathrm{inv}}$ is trivial. In particular, each c_j is σ-invariant. Write $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the -1 eigenspace of σ. Then $\mathfrak{p}$ is identified with the tangent space to M at the identity coset, and the multi-vector field defined by c_j is given as the projection of c_j onto $\wedge\mathfrak{p}$. But since c_j has odd degree, and σ acts as -1 on $\wedge^{\mathrm{odd}}\mathfrak{p}$, it follows that the vector field is just 0. That is, $\iota(c_j) = 0$ for all j. We conclude that any invariant differential form on M is an *equivariant* cocycle for the small Cartan model. Applying the operator $e^{\iota(f)}$, we directly get the equivariant extension for the standard (large) Cartan model.

4.3. **Dependence on f.** Returning to the setting of Theorem 4.2, it is natural to ask to what extent the isomorphism $\tilde{H}_\mathfrak{g}(\mathcal{M}) \to H_\mathfrak{g}(\mathcal{M})$ depends on the choice of f. Recall that any two solutions f_0, f_1 of (17) are gauge equivalent, up to addition of an even element in $\mathfrak{l} = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g})^-_{\mathrm{inv}}$. The action of $(\wedge\mathfrak{g})^-_{\mathrm{inv}}$ on the small Cartan model is homotopic to the trivial action: Indeed, let h_i denote the derivation of $(S\mathfrak{g}^*)_{\mathrm{inv}}$ given by $h_i(p^j) = \delta_i^j$. Then

$$[\tilde{\mathrm{d}}_\mathfrak{g}, h_j \otimes 1] = -\iota(c_j),$$

showing that $\iota(c_j)$ is homotopic to 0. It is therefore sufficient to consider the case that f_0, f_1 are gauge equivalent:

$$f_1 = \exp(s_1).f_0 = e^{\mathrm{ad}_{s_1}} f_0 - j^R(\mathrm{ad}_{s_1})\partial s_1$$

for an odd element $s_1 \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$. (Here exp is the Lie algebra exponential, defined as in Section 3.3.) Let $s(t) = ts_1$ for $t \in \mathbb{F}$. Then f_0, f_1 belong to a family of solutions $f(t)$ of (17), depending *polynomially* on t (since s_1 is nilpotent):

$$f(t) = \exp(s(t)).f_0 = e^{\mathrm{ad}_{s(t)}} f_0 - j^R(\mathrm{ad}_{s(t)})\partial s(t).$$

We have,

$$\frac{df}{dt} + \partial s + [f,s]_{\wedge\mathfrak{g}} = 0. \tag{35}$$

Now let $\mathcal{M}$ be any $\mathfrak{g}$-differential space.

Proposition 4.5. *Let $\Phi(t) = e^{\iota(f(t))} \colon \tilde{C}_{\mathfrak{g}}(\mathcal{M}) \to C_{\mathfrak{g}}(\mathcal{M})$ be the family of cochain maps defined by $f(t)$. Then the family of operators*

$$H(t) = \Phi(t) \circ \iota(s(t)) : (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{M}_{\mathrm{inv}} \to (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}},$$

satisfies, $\frac{d\Phi}{dt} = H \circ \tilde{\mathrm{d}}_{\mathfrak{g}} + \mathrm{d}_{\mathfrak{g}} \circ H$. Thus, all $\Phi(t)$ are homotopic as homomorphisms of $(S\mathfrak{g}^*)_{\mathrm{inv}}$-differential modules.

Note that $\Phi(t)$ and $H(t)$ depend polynomially on t.

Proof. We thank the referee for the following simplified version of our original argument. By (34), we have

$$\mathrm{d}_{\mathfrak{g}} \circ \Phi = \Phi \circ \big(\tilde{\mathrm{d}}_{\mathfrak{g}} + \sum_a \iota(\iota^*(e^a)f) L^{\mathcal{M}}(e_a)\big).$$

Furthermore, the commutator of $\sum_a \iota(\iota^*(e^a)f)L^{\mathcal{M}}(e_a)$ with $\iota(s)$ gives $-\iota([f,s]_{\wedge\mathfrak{g}})$. Hence, using Lemma 2.1 and Equation (35),

$$\begin{aligned} H \circ \tilde{\mathrm{d}}_{\mathfrak{g}} + \mathrm{d}_{\mathfrak{g}} \circ H &= \Phi \circ \iota(s) \circ \tilde{\mathrm{d}}_{\mathfrak{g}} + \mathrm{d}_{\mathfrak{g}} \circ \Phi \circ \iota(s) \\ &= \Phi \circ \big([\tilde{\mathrm{d}}_{\mathfrak{g}}, \iota(s)] + \sum_a \iota(\iota^*(e^a)f) \circ L^{\mathcal{M}}(e_a) \circ \iota(s)\big) \\ &= \Phi \circ \big([\tilde{\mathrm{d}}_{\mathfrak{g}}, \iota(s)] - \iota([f,s]_{\wedge\mathfrak{g}})\big) + \ldots \\ &= -\Phi \circ \iota(\partial s + [f,s]_{\wedge\mathfrak{g}}) + \ldots \\ &= \Phi \circ \iota\big(\frac{df}{dt}\big) + \ldots \\ &= \frac{d\Phi}{dt} + \ldots, \end{aligned}$$

where the dots indicate terms vanishing on $\tilde{C}$. ∎

To summarize, we have shown:

Theorem 4.6. *The homomorphism of differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules $\Phi \colon \tilde{C}_{\mathfrak{g}}(\mathcal{M}) \to C_{\mathfrak{g}}(\mathcal{M})$ is independent of the solution f of (17), up to $(S\mathfrak{g}^*)_{\mathrm{inv}}$-equivariant homotopy. In particular, the induced map in cohomology does not depend on the choice of f.*

5. The Chevalley-Koszul complex

5.1. **$\mathfrak{g}$-differential $W\mathfrak{g}$-modules.** A $\mathfrak{g}$-*differential algebra* is a graded associative algebra $\mathcal{A}$, together with the structure of a $\mathfrak{g}$-differential space in such a way that the operators $\mathrm{d}, L(\xi), \iota(\xi)$ are derivations for the product structure. A $\mathfrak{g}$-*differential module* for $\mathcal{A}$ is a $\mathfrak{g}$-differential space $\mathcal{N}$, with an $\mathcal{A}$-module structure such that the action map $\mathcal{A} \otimes \mathcal{N} \to \mathcal{N}$ is a homomorphism of $\mathfrak{g}$-differential spaces. Of particular importance is the case where $\mathcal{A} = W\mathfrak{g}$ is the Weil algebra. If $\mathcal{N}$ is $\mathfrak{g}$-differential $W\mathfrak{g}$-module, then the basic subcomplex $\mathcal{N}_{\mathrm{basic}}$ is naturally a differential $(W\mathfrak{g})_{\mathrm{basic}} = (S\mathfrak{g}^*)_{\mathrm{inv}}$-module. $\mathfrak{g}$-differential $W\mathfrak{g}$-modules were studied by Guillemin-Sternberg in [9], under the name 'W^*-modules'.

Examples 5.1. (a) Suppose $\mathcal{N}$ is a commutative $\mathfrak{g}$-differential algebra, equipped with an algebraic connection $\theta\colon \mathfrak{g}^* \to \mathcal{N}$ in the sense of Cartan [3]. Recall that the algebraic Chern-Weil homomorphism $CW_\theta\colon W\mathfrak{g} \to \mathcal{N}$ is the unique homomorphism of $\mathfrak{g}$-differential algebras extending the map θ on $\wedge^1\mathfrak{g}^* \subset W\mathfrak{g}$. Clearly, CW_θ gives $\mathcal{N}$ the structure of a $\mathfrak{g}$-differential $W\mathfrak{g}$-module.

(b) For any $\mathfrak{g}$-differential space $\mathcal{M}$, the tensor product $\mathcal{N} = W\mathfrak{g} \otimes \mathcal{M}$ is a $\mathfrak{g}$-differential $W\mathfrak{g}$-module in the obvious way. Note that since $\mathbb{F} \hookrightarrow W\mathfrak{g}$ is a homotopy equivalence, the inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$ is a quasi-isomorphism of $\mathfrak{g}$-differential spaces.

As before, we let y^a and v^a denote the generators of $W\mathfrak{g}$ for a given basis of $\mathfrak{g}$. For any $\mathfrak{g}$-differential $W\mathfrak{g}$-module $\mathcal{N}$, there is a canonical *horizontal projection* operator

$$P_{\mathrm{hor}} = \prod_a \iota^{\mathcal{N}}(e_a) y^a \colon \mathcal{N} \to \mathcal{N}_{\mathrm{hor}}.$$

The operator P_{hor} is a $\mathfrak{g}$-equivariant morphism of modules over $S\mathfrak{g}^*$.

We will need the following fact:

Theorem 5.2 (Cartan [2], see also [9, 18])**.** *Let $\mathcal{N}$ be a $\mathfrak{g}$-differential $W\mathfrak{g}$-module. The map $S\mathfrak{g}^* \otimes \mathcal{N} \to \mathcal{N}_{\mathrm{hor}},\ p \otimes x \mapsto P_{\mathrm{hor}}(p.x)$ restricts to a* cochain map*,*

$$C_{\mathfrak{g}}(\mathcal{N}) = (S\mathfrak{g}^* \otimes \mathcal{N})_{\mathrm{inv}} \to \mathcal{N}_{\mathrm{basic}}. \tag{36}$$

This cochain map is an $(S\mathfrak{g}^*)_{\mathrm{inv}}$-equivariant projection, and is homotopy inverse (in the category of $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules) to the natural inclusion $\mathcal{N}_{\mathrm{basic}} \to (S\mathfrak{g}^* \otimes \mathcal{N})_{\mathrm{inv}}$.

It is useful to note that any $\mathfrak{g}$-differential $W\mathfrak{g}$-module $\mathcal{N}$ may be written as a tensor product of $\wedge\mathfrak{g}^*$ with an $S\mathfrak{g}^*$-module. Let $\mathrm{d}_{\mathrm{hor}} = P_{\mathrm{hor}} \circ \mathrm{d}$ is the *covariant derivative* on $\mathcal{N}$, given on horizontal elements $x \in \mathcal{N}_{\mathrm{hor}}$ by $\mathrm{d}_{\mathrm{hor}} x = (\mathrm{d} - \sum_a y^a\, L^{\mathcal{N}}(e_a))x$.

Proposition 5.3. [7, Section 8.7] *The action of $\wedge\mathfrak{g}^* \subset W\mathfrak{g}$ on $\mathcal{N}$ restricts to a $\mathfrak{g}$-equivariant isomorphism,*

$$\mathcal{N}_{\mathrm{hor}} \otimes \wedge\mathfrak{g}^* \to \mathcal{N},\ \ x \otimes \eta \mapsto (-1)^{|\eta||x|}\eta.x. \tag{37}$$

This isomorphism intertwines the action of $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ on $\mathcal{N}_{\mathrm{hor}} \otimes \wedge\mathfrak{g}^*$ with the given action on $\mathcal{N}$. It induces the following differential on $\mathcal{N}_{\mathrm{hor}} \otimes \wedge\mathfrak{g}^*$,

$$-1 \otimes \mathrm{d}^\wedge + \mathrm{d}_{\mathrm{hor}} \otimes 1 + \sum_a v^a \otimes \iota^\wedge(e_a) + \sum_a (1 \otimes y^a) L(e_a). \tag{38}$$

Here $L(\xi) = L^{\mathcal{N}}(\xi) \otimes 1 + 1 \otimes L^\wedge(\xi)$ are the Lie derivatives for the diagonal action.

Note that for $\mathcal{N} = W\mathfrak{g}$, one recovers the formula (13) for the Weil differential.

Proof. Define two $\mathfrak{g}$-equivariant degree 0 endomorphisms of $\mathcal{N} \otimes \wedge\mathfrak{g}^*$,

$$\alpha = \sum_a \iota^{\mathcal{N}}(e_a) \otimes y^a, \quad \beta = \sum_a y^a \otimes \iota^\wedge(e_a).$$

Since α, β are nilpotent, their exponentials are well-defined automorphisms of degree 0. An elementary calculation shows,

$$\mathrm{Ad}(e^\alpha)(1 \otimes \iota^\wedge(\xi)) = \iota^{\mathcal{N}}(\xi) \otimes 1 + 1 \otimes \iota^\wedge(\xi) = \mathrm{Ad}(e^{-\beta})(\iota^{\mathcal{N}}(\xi) \otimes 1)$$

Thinking of $\mathcal{N}$ as the joint kernel of the operators $1\otimes\iota^\wedge(\xi)$ on $\mathcal{N}\otimes\wedge\mathfrak{g}^*$, this gives isomorphisms,

$$\mathcal{N}_{\mathrm{hor}}\otimes\wedge\mathfrak{g}^* \xrightarrow{e^\beta} (\mathcal{N}\otimes\wedge\mathfrak{g}^*)_{\mathrm{hor}} \xrightarrow{e^{-\alpha}} \mathcal{N}. \tag{39}$$

On $(\mathcal{N}\otimes\wedge\mathfrak{g}^*)_{\mathrm{hor}}$, the operator $e^{-\alpha}=\prod_a(1-\iota^{\mathcal{N}}(e_a)\otimes y^a)$ coincides with

$$1\otimes\prod_a(1-y^a\iota^\wedge(e_a))=1\otimes\prod_a\iota^\wedge(e_a)y^a=1\otimes P^\wedge_{\mathrm{hor}},$$

where $P_{\mathrm{hor}}\colon\wedge\mathfrak{g}^*\to\mathbb{F}$ is the horizontal projection map for $\wedge\mathfrak{g}^*$. But a moment's reflection shows that $(1\otimes P^\wedge_{\mathrm{hor}})\circ e^\beta\colon\mathcal{N}_{\mathrm{hor}}\otimes\wedge\mathfrak{g}^*\to\mathcal{N}$ is exactly the map, $x\otimes\eta\mapsto(-1)^{|\eta||x|}\eta.x$. Clearly, this map is compatible with the $W\mathfrak{g}$-module structures.

Now let d' be the differential on $\mathcal{N}_{\mathrm{hor}}\otimes\wedge\mathfrak{g}^*$, induced via (39) from the differential on $\mathcal{N}$, and let d'' be the derivation (38). To show $\mathrm{d}'=\mathrm{d}''$, it suffices to prove that the two derivations agree on $\mathcal{N}_{\mathrm{hor}}\otimes 1$, and that $[\mathrm{d}',1\otimes y^a]=[\mathrm{d}'',1\otimes y^a]$.

Let $x\in\mathcal{N}_{\mathrm{hor}}$. Since (39) takes $x\otimes 1$ to x, $\mathrm{d}'(x\otimes 1)$ is the inverse image of $\mathrm{d}^{\mathcal{N}}x=\mathrm{d}^{\mathcal{N}}_{\mathrm{hor}}x+\sum_a y^a.L^{\mathcal{N}}(e_a)x$. Thus

$$\mathrm{d}'(x\otimes 1)=\mathrm{d}^{\mathcal{N}}_{\mathrm{hor}}x\otimes 1+(-1)^{|x|}\sum_a L^{\mathcal{N}}(e_a)x\otimes y^a=\mathrm{d}''(x\otimes 1). \tag{40}$$

We next observe that $[\mathrm{d}^{\mathcal{N}},y^a]=\mathrm{d}^W y^a=v^a+\mathrm{d}^\wedge y^a$ since $\mathcal{N}$ is a differential $W\mathfrak{g}$-module. Hence,

$$[\mathrm{d}',1\otimes y^a]=v^a\otimes 1+1\otimes\mathrm{d}^\wedge y^a=[\mathrm{d}'',1\otimes y^a]. \tag{41}$$

□

5.2. **The Chevalley-Koszul complex.** The *Chevalley-Koszul complex* of a $\mathfrak{g}$-differential $W\mathfrak{g}$-module $\mathcal{N}$ is the differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-module,

$$\mathcal{N}_{\mathrm{basic}}\otimes(\wedge\mathfrak{g}^*)_{\mathrm{inv}},\quad \mathrm{d}\otimes 1+\sum_j p^j\otimes\iota^\wedge(c_j). \tag{42}$$

Remark 5.4. If $\mathfrak{g}$ is Abelian, (42) is the same as $(\mathcal{N}_{\mathrm{hor}}\otimes\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ with the differential (38).

Let $f\in(S\mathfrak{g}^*\otimes(\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ be a degree 0 solution of the Maurer-Cartan equation (17). Given a $\mathfrak{g}$-differential $W\mathfrak{g}$-module $\mathcal{N}$, let $\iota^{\mathcal{N}}(f)\in\mathrm{End}(\mathcal{N})$ be the nilpotent operator, where the $S\mathfrak{g}^*$-factor acts via the $W\mathfrak{g}$-module structure, and the $\wedge\mathfrak{g}$ factor acts by contraction. (In the special case $\mathcal{N}=W\mathfrak{g}$, we write $\iota^W(f)$.) Let $\alpha,\beta\in\mathrm{End}(\mathcal{N}_{\mathrm{inv}}\otimes(\wedge\mathfrak{g}^*)_{\mathrm{inv}})$ be the nilpotent endomorphisms of degree 0,

$$\alpha=\sum_j\iota(c_j)\otimes c^j,\ \ \beta=\sum_j c^j\otimes\iota(c_j).$$

Theorem 5.5. *The Chevalley-Koszul complex of $\mathcal{N}$ is homotopy equivalent, as a differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-module, to the invariant subcomplex of $\mathcal{N}$. In more detail,*

(a) *There is a homomorphism of differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules,*

$$\Psi\colon\mathcal{N}_{\mathrm{basic}}\otimes(\wedge\mathfrak{g}^*)_{\mathrm{inv}}\to\mathcal{N}_{\mathrm{inv}},\ \ z\otimes\eta\mapsto(-1)^{|\eta||z|}(e^{\iota^W(f)}\eta).z.$$

(b) *There is a homomorphism of differential* $(\wedge\mathfrak{g})_{\mathrm{inv}}$*-modules,*

$$\Upsilon\colon \mathcal{N}_{\mathrm{inv}} \to \mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}},\ z \mapsto (P_{\mathrm{hor}} \otimes 1) \circ e^{-\alpha}(e^{-\iota^{\mathcal{N}}(f)} z \otimes 1).$$

(c) *The composition* $\Upsilon \circ \Psi$ *is equal to the identity, while* $\Psi \circ \Upsilon$ *is homotopic to the identity by a* $(\wedge\mathfrak{g})_{\mathrm{inv}}$*-homotopy.*

Proof. (a) Write $\iota\colon S\mathfrak{g}^* \otimes \wedge\mathfrak{g} \to \mathrm{End}(\mathcal{N} \otimes \wedge\mathfrak{g}^*)$, where the first factor acts on $\mathcal{N}$ via the $W\mathfrak{g}$-module structure, and the second factor acts by contraction. The map Ψ can be written as a composition of two homomorphisms of $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules,

$$\mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \xrightarrow{e^{\iota(f)}} (\mathcal{N}_{\mathrm{hor}} \otimes (\wedge\mathfrak{g}^*))_{\mathrm{inv}} \to \mathcal{N}_{\mathrm{inv}},$$

where the second map is an isomorphism given by (37). Notice that the last term in the formula (38) on $\mathcal{N}_{\mathrm{hor}} \otimes (\wedge\mathfrak{g}^*)$ vanishes on invariants. Hence, the differential on $(\mathcal{N}_{\mathrm{hor}} \otimes (\wedge\mathfrak{g}^*))_{\mathrm{inv}}$ is

$$-1 \otimes \mathrm{d}^\wedge + \mathrm{d}_{\mathrm{hor}} \otimes 1 + \sum_a v^a \otimes \iota^\wedge(e_a).$$

Conjugation of this result by the automorphism $e^{-\iota(f)}$ gives,

$$-1 \otimes \mathrm{d}^\wedge + \mathrm{d}_{\mathrm{hor}} \otimes 1 + \sum_j p^j \otimes \iota^\wedge(c_j) + \sum_a \iota(\iota^*(e^a) f)\, L^{\mathcal{N}}(e_a). \tag{43}$$

On the subspace $\mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$, both $L^{\mathcal{N}}(e_a)$ and $1 \otimes \mathrm{d}^\wedge$ vanish while $\mathrm{d}_{\mathrm{hor}} \otimes 1$ coincides with $\mathrm{d} \otimes 1$. This shows that Ψ is a cochain map, proving (a). As a preparation for (c), let us also remark that Ψ is a homotopy equivalence of differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules, by an argument parallel to the proof of Theorem 4.2. That is, starting with the operator $h = 1 \otimes \sum_a \iota(B^\sharp(e^a)) L(e_a)$ one constructs a homotopy on $(\mathcal{N}_{\mathrm{hor}} \otimes \wedge\mathfrak{g}^*)_{\mathrm{inv}}$ between the identity and some projection onto $\mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$.

(b) Consider the tensor product $K(\mathcal{P}) \otimes \mathcal{N}_{\mathrm{inv}}$ with differential, $\mathrm{d}^K \otimes 1 + 1 \otimes \mathrm{d}^{\mathcal{N}}$ and with the $\wedge\mathcal{P}$-module structure given by $\iota(c_j) \otimes 1 + 1 \otimes \iota(c_j)$. Clearly, the inclusion $z \mapsto 1 \otimes z$ of $\mathcal{N}_{\mathrm{inv}}$ is a homomorphism of differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules. Under the automorphism $\exp(\beta)$ of $K(\mathcal{P}) \otimes \mathcal{N}_{\mathrm{inv}}$,

$$\begin{aligned} e^\beta \circ (\iota^K(c_k) \otimes 1 + 1 \otimes \iota^{\mathcal{N}}(c_k)) \circ e^{-\beta} &= \iota^K(c_k) \otimes 1, \\ e^\beta \circ (\mathrm{d}^K \otimes 1 + 1 \otimes \mathrm{d}^{\mathcal{N}}) \circ e^{-\beta} &= \mathrm{d}^K \otimes 1 + 1 \otimes \mathrm{d}^{\mathcal{N}} - \sum_j p^j \otimes \iota(c_j). \end{aligned}$$

Writing $K(\mathcal{P}) = (S\mathfrak{g}^*)_{\mathrm{inv}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$, and re-arranging the factors, we hence obtain an isomorphism of $K(\mathcal{P}) \otimes \mathcal{N}_{\mathrm{inv}}$ with $\tilde{C}_{\mathfrak{g}}(\mathcal{N}) \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$, with differential

$$\tilde{\mathrm{d}}_{\mathfrak{g}} \otimes 1 + \sum_j p^j \otimes \iota^\wedge(c_j),$$

and with $(\wedge\mathfrak{g})_{\mathrm{inv}}$-module structure given by by contractions on the first factor. By Theorem 4.2 together with Cartan's Theorem 5.2, there is a homotopy equivalence of differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules,

$$(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{N}_{\mathrm{inv}} = \tilde{C}_{\mathfrak{g}}(\mathcal{N}) \to C_{\mathfrak{g}}(\mathcal{N}) \to \mathcal{N}_{\mathrm{basic}},\quad p \otimes z \mapsto p.P_{\mathrm{hor}}(e^{\iota^W(f)}.z).$$

The resulting homomorphism of differential $(\wedge\mathfrak{g})_{\mathrm{inv}}$-modules,

$$\mathcal{N}_{\mathrm{inv}} \to (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \otimes \big((S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{N}_{\mathrm{inv}}\big) \to (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{N}_{\mathrm{basic}}$$

is exactly our map Υ.

(c) Let $z \otimes \eta \in \mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. We compute,

$$\begin{aligned}\Upsilon(\Psi(z \otimes \eta)) &= (-1)^{|z||\eta|}\Upsilon\big((e^{\iota^W(f)}\eta).z\big)\\ &= (-1)^{|z||\eta|}(P_{\mathrm{hor}} \otimes 1) \circ e^{-\alpha} e^{-\iota^{\mathcal{N}}(f)\otimes 1}((e^{\iota^W(f)}\eta).z \otimes 1)\\ &= (-1)^{|z||\eta|}(P_{\mathrm{hor}} \otimes 1) \circ e^{-\alpha}(\eta.z \otimes 1)\\ &= z \otimes \eta.\end{aligned}$$

Thus $\Upsilon \circ \Psi = \mathrm{Id}$, while the opposite composition $\Pi = \Psi \circ \Upsilon$ is a projection. As remarked in (a) above, there exists a $(\wedge\mathfrak{g})_{\mathrm{inv}}$-homotopy operator H_1 between I and *some* projection operator Π_1 onto $\mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$. Lemma 5.6 below shows how to obtain from this a $(\wedge\mathfrak{g})_{\mathrm{inv}}$-homotopy operator H between I and Π. □

Lemma 5.6. *Let (C, d) be an $\mathcal{A}$-differential space, where $\mathcal{A}$ is some graded algebra (with trivial differential). Suppose $H_1 \colon C \to C$ is an $\mathcal{A}$-equivariant homotopy operator between the identity and some $\mathcal{A}$-equivariant projection $\Pi_1 \colon C \to C$ onto a differential subspace $C' \subset C$. Assume that some cochain map $\Pi \colon C \to C$ is another $\mathcal{A}$-equivariant projection onto C'. Then*

$$H = (I - \Pi_1 - \Pi)H_1(I - \Pi_1 - \Pi)$$

is an $\mathcal{A}$-equivariant homotopy between I and Π.

Proof. This follows by straightforward calculation, using $\Pi_1\Pi = \Pi$ and $\Pi\,\Pi_1 = \Pi_1$:

$$\begin{aligned}[\mathrm{d}, H] &= (I - \Pi_1 - \Pi)[\mathrm{d}, H_1](I - \Pi_1 - \Pi)\\ &= (I - \Pi_1 - \Pi)(I - \Pi_1)(I - \Pi_1 - \Pi) = I - \Pi.\end{aligned}$$

□

Remark 5.7. Suppose f_0, f_1 are two solutions of (17). Then, the corresponding cochain maps from $(\wedge\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{N}_{\mathrm{basic}}$ into $\mathcal{N}_{\mathrm{inv}}$ are homotopic, by a homotopy operator which is compatible with the $(\wedge\mathfrak{g}^*)_{\mathrm{inv}}$-module structure. The proof is parallel to that of Proposition 4.5.

Remark 5.8. Recall that in 3.3, we described a homomorphism of differential graded algebras $\Phi' \colon K(P) \to (W\mathfrak{g})_{\mathrm{inv}}$, depending on the choice of distinguished cochains of transgression. Thinking of $(\wedge\mathfrak{g}^*)_{\mathrm{inv}} = \wedge P^*$ as a subalgebra of $K(P)$, this gives a cochain map,

$$\Psi' \colon \mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}} \xrightarrow{1\otimes\Phi'} \mathcal{N}_{\mathrm{basic}} \otimes (W\mathfrak{g})_{\mathrm{inv}} \to \mathcal{N}_{\mathrm{inv}},$$

where the last map is $x \otimes w \mapsto (-1)^{|x||w|}w.x$. This map is described in [7, page 364] under the name *Chevalley homomorphism*. Assuming that the complex $\mathcal{N}$ is bounded below, an easy spectral sequence argument shows that Ψ' induces an isomorphism in cohomology [7, page 365]. In [17], it is claimed that Ψ' is a homomorphism of $\wedge P$-modules, but this is false (see Remark 3.3).

The fact that the complex $\mathcal{N}_{\mathrm{basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathrm{inv}}$ computes the cohomology of $\mathcal{N}_{\mathrm{inv}}$ (hence of $\mathcal{N}$, since $\mathfrak{g}$ is reductive) goes back to Chevalley and Koszul. See the article of Koszul [15] in the 'Colloque de topologie' (reproduced in [16]). As a special case of this result, one obtains the

de Rham cohomology of the total space of a principal G-bundle $P \to B$ (with G a compact Lie group) as the cohomology of a complex $\Omega(B) \otimes (\wedge \mathfrak{g}^*)_{\mathrm{inv}}$, where the complex $\Omega(B)$ of differential forms on the base is viewed as an $(S\mathfrak{g}^*)_{\mathrm{inv}}$-module by the Chern-Weil homomorphism.

As explained in [6, 17], Theorems 5.5 and 4.2 are related by Koszul duality. We outline this argument in Appendix B. In Appendix A we indicate a common generalization of the two theorems.

6. Lie algebra homomorphisms

In this Section, we will address the functoriality properties of the small Cartan model and of the Chevalley-Koszul model under homomorphisms of reductive Lie algebras. In particular, taking $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ the diagonal embedding, this will also lead to a description of product structures for the two models.

6.1. **Lie algebra homomorphisms.** Suppose $\mathfrak{h}$ is another reductive Lie algebra, with $\mathcal{Q} \subset (\wedge \mathfrak{h})_{\mathfrak{h}\text{-inv}}$ as its space of primitive elements, and that $\phi \colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. Then ϕ extends to an algebra homomorphism $\phi \colon \wedge \mathfrak{g} \to \wedge \mathfrak{h}$, compatible with the boundary operator and with the Schouten bracket. Furthermore, the dual map restricts to a linear map

$$\phi^* \colon (\wedge \mathfrak{h}^*)_{\mathfrak{h}\text{-inv}} \to (\wedge \mathfrak{g}^*)_{\mathfrak{g}\text{-inv}},$$

which is a homomorphism of Hopf algebras [7, Section 5.17]. Hence it restricts to a linear map,

$$\phi^* \colon \mathcal{Q}^* \to \mathcal{P}^*.$$

Let $d_l \in \mathcal{Q}$ and $q^l \in \tilde{\mathcal{Q}}^*$ by dual (homogeneous) bases.

Proposition 6.1. *There exists a degree* 0 *solution* $u \in (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge \mathfrak{h})^-_{\mathfrak{g}\text{-inv}}$ *of the equation,*

$$\partial u + \tfrac{1}{2}[u,u]_{\wedge \mathfrak{h}} = \sum_l \phi^*(q^l) \otimes d_l - \sum_j p^j \otimes \phi(c_j). \tag{44}$$

Proof. We will apply Theorem 3.8 to the setting,

$$\begin{aligned}
\mathfrak{k} &= \bigoplus_{i \le 0} \mathfrak{k}^i, \quad \mathfrak{k}^i = (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge^{1-i} \mathfrak{h})_{\mathfrak{g}\text{-inv}}, \\
\mathfrak{l} &= \bigoplus_{i \le 0} \mathfrak{l}^i, \quad \mathfrak{l}^i = (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge^{1-i} \mathfrak{h})_{\mathfrak{h}\text{-inv}}, \\
X &= \sum_l \phi^*(q^l) \otimes d_l - \sum_j p^j \otimes \phi(c_j).
\end{aligned}$$

Here the bracket $[\cdot,\cdot]_{\mathfrak{k}}$ and differential ∂ on $\mathfrak{k}$ are induced from the Schouten bracket and differential on $(\wedge \mathfrak{h})_{\mathfrak{g}\text{-}\mathrm{inv}} \subset \wedge \mathfrak{h}$. Observe that $\mathfrak{l}$ is contained in the center of $\mathfrak{k}$, since $(\wedge \mathfrak{h})_{\mathfrak{h}\text{-}\mathrm{inv}}$ is the center of $\wedge \mathfrak{h}$. The differential vanishes on $\mathfrak{l}$, and by Hodge theory for $\wedge \mathfrak{h}$ the inclusion $\mathfrak{l} \hookrightarrow \mathfrak{k}$ induces an isomorphism in cohomology. Finally X is a cocycle (since $\partial q^l = 0$ and $\partial c^j = 0$), and is contained in the center since elements in $\phi(\wedge \mathfrak{g})$ Schouten commute with $\mathfrak{g}$-invariant elements in $\wedge \mathfrak{h}$. Hence all assumptions of Theorem 3.8 are satisfied, and we obtain a solution u of (44)

modulo $\mathfrak{l}$, (Using Hodge theory on $\wedge\mathfrak{h}$, we obtain in fact a *canonical* solution u of (total) degree 0.) Theorem 3.8 says furthermore that the 'error term'

$$Y := \partial u + \tfrac{1}{2}[u,u]_{\wedge\mathfrak{g}} + \sum_l \phi^*(q^l)\otimes d_l - \sum_j p^j \otimes \phi(c_j) \in \mathfrak{l}$$

does not depend on the choice of u. It remains to show that in fact $Y = 0$.

Let $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathfrak{g}\text{-inv}}$ be an even element solving (17). Using again that elements in $\phi(\wedge\mathfrak{g})$ and $(\wedge\mathfrak{h})_{\mathfrak{g}\text{-inv}}$ commute under the Schouten bracket, we have

$$[(1\otimes\phi)(f), u]_{\wedge\mathfrak{h}} = 0.$$

It follows that

$$\tilde{f}_1 := (1\otimes\phi)(f) + u \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{h})^-)_{\mathfrak{g}\text{-inv}}$$

satisfies the equation

$$\partial\tilde{f}_1 + \tfrac{1}{2}[\tilde{f}_1,\tilde{f}_1]_{\wedge\mathfrak{h}} = \sum_l \phi^*(q^l)\otimes d_l - \sum_a v^a\otimes\phi(e_a) + Y. \tag{45}$$

On the other hand, let $g \in (S\mathfrak{h}^* \otimes (\wedge\mathfrak{h})^-)_{\mathfrak{h}\text{-inv}}$ be an even solution of the analogue of (17) for the Lie algebra $\mathfrak{h}$. Then $\tilde{f}_0 = (\phi^*\otimes 1)(g)$ solves a similar equation, but with Y replaced by 0:

$$\partial\tilde{f}_0 + \tfrac{1}{2}[\tilde{f}_0,\tilde{f}_0]_{\wedge\mathfrak{h}} = \sum_l \phi^*(q^l)\otimes d_l - \sum_a v^a\otimes\phi(e_a). \tag{46}$$

(Here we used that the elementary fact that the image of the canonical element $\sum_a v^a\otimes e_a \in S\mathfrak{g}^*\otimes\wedge\mathfrak{g}$ under $1\otimes\phi$ coincides with the image of the corresponding element of $S\mathfrak{h}^*\otimes\wedge\mathfrak{h}$ under the map $\phi^*\otimes 1$.) By the uniqueness part of Theorem 3.8, applied to the situation

$$\mathfrak{k} = \bigoplus_{i\le 0}\mathfrak{k}^i,\quad \mathfrak{k}^i = (S\mathfrak{g}^*\otimes(\wedge\mathfrak{h}))_{\mathfrak{h}\text{-inv}},$$
$$\mathfrak{l} = \bigoplus_{i\le 0}\mathfrak{l}^i,\quad \mathfrak{l}^i = (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}}\otimes(\wedge^{1-i}\mathfrak{h})_{\mathfrak{h}\text{-inv}},$$
$$X = \sum_l \phi^*(q^l)\otimes d_l - \sum_a v^a\otimes\phi(e_a),$$

this shows $Y = 0$. $\square$

Remark 6.2. If the Lie algebra $\mathfrak{g}$ is *Abelian*, one may simply take $u = (\phi^*\otimes 1)(g)$ where $g \in (S\mathfrak{h}^*\otimes(\wedge\mathfrak{h})^-)_{\mathfrak{h}\text{-inv}}$ solves (22). For instance, let $\mathfrak{h}$ be the 3-dimensional Lie algebra from Example 3.13 (denoted $\mathfrak{g}$ in that example), and let $\mathfrak{g}\subset\mathfrak{h}$ be the inclusion of the 1-dimensional Lie subalgebra spanned by e_3. Then $u = v^3\otimes(e_1\wedge e_2)$ is a solution of (44).

As a special case of Proposition 6.1, consider the diagonal inclusion $\mathrm{diag}\colon \mathfrak{g}\to\mathfrak{g}\oplus\mathfrak{g}$, and its extension to the exterior algebra, $\mathrm{diag}\colon \wedge\mathfrak{g}\to\wedge\mathfrak{g}\otimes\wedge\mathfrak{g}$. The dual map diag^* is just the product map for $\wedge\mathfrak{g}^*$. Hence Proposition 6.1 defines a solution $u\in(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}}\otimes(\wedge\mathfrak{g}\otimes\wedge\mathfrak{g})_{\mathfrak{g}\text{-inv}}$ of the equation,

$$\partial u + \tfrac{1}{2}[u,u]_{\wedge(\mathfrak{g}\oplus\mathfrak{g})} = \sum_j p^j\otimes(\mathrm{diag}(c_j) - \Delta(c_j)) \tag{47}$$

where Δ is the coproduct on $(\wedge\mathfrak{g})_{\mathrm{inv}}$, i.e. $\Delta(c_j) = c_j\otimes 1 + 1\otimes c_j$.

Example 6.3. Consider $\mathfrak{g}$ as in Example 3.13, with $p = \sum_a v^a v^a$. Write $e_a^{(1)}$ (resp. $e_a^{(2)}$) for the basis vectors in the first (resp. second) copy of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}$. Then

$$u = p \otimes (e_1^{(1)} \wedge e_2^{(1)} \wedge e_1^{(2)} \wedge e_2^{(2)} + \dots)$$

(where the dots indicate a sum over cyclic permutations over the lower indices $1, 2, 3$) solves Equation (47). Note that $[u, u]_{\wedge(\mathfrak{g}\oplus\mathfrak{g})} = 0$ in this case.

6.2. **Small Cartan complex.** Let $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of reductive Lie algebras, and $\mathcal{M}$ be an $\mathfrak{h}$-differential space. The natural map

$$C_{\mathfrak{h}}(\mathcal{M}) = (S\mathfrak{h}^* \otimes \mathcal{M})_{\mathfrak{h}\text{-inv}} \to C_{\mathfrak{g}}(\mathcal{M}) = (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathfrak{g}\text{-inv}}$$

is a cochain map, inducing a map in cohomology, $H_{\mathfrak{h}}(\mathcal{M}) \to H_{\mathfrak{g}}(\mathcal{M})$. We will now realize this map in terms of the small Cartan models.

Theorem 6.4. *Let $u \in (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge\mathfrak{h})_{\mathfrak{g}\text{-inv}}$ be a solution of (44). Then the operator*

$$\Psi = e^{\iota(u)} \circ (\phi^* \otimes 1)\colon (S\mathfrak{h}^*)_{\mathfrak{h}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{h}\text{-inv}} \to (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{g}\text{-inv}}$$

is a cochain map. The resulting diagram of differential $(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}}$-modules,

$$\begin{tikzcd} (S\mathfrak{h}^*)_{\mathfrak{h}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{h}\text{-inv}} \arrow[r] \arrow[d] & (S\mathfrak{h}^* \otimes \mathcal{M})_{\mathfrak{h}\text{-inv}} \arrow[d] \\ (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{g}\text{-inv}} \arrow[r] & (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathfrak{g}\text{-inv}} \end{tikzcd} \tag{48}$$

commutes up to $(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}}$-equivariant homotopy. In particular, the map $\tilde{H}_{\mathfrak{h}}(\mathcal{M}) \to \tilde{H}_{\mathfrak{g}}(\mathcal{M})$ defined by Ψ agrees with the natural map $H_{\mathfrak{h}}(\mathcal{M}) \to H_{\mathfrak{g}}(\mathcal{M})$, under the identification $\tilde{H}_{\mathfrak{g}}(\mathcal{M}) = H_{\mathfrak{g}}(\mathcal{M})$, $\tilde{H}_{\mathfrak{h}}(\mathcal{M}) = H_{\mathfrak{h}}(\mathcal{M})$ from Theorem 4.2.

Proof. Since the image of $(S\mathfrak{h}^*)_{\mathfrak{h}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{h}\text{-inv}}$ under the map $(\phi^* \otimes 1)$ is contained in $(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{h}\text{-inv}}$, (10) and (44) show that

$$(1 \otimes \mathrm{d}) \circ \Psi = \Psi \circ \big(1 \otimes \mathrm{d} - \sum_l \phi^*(q^l) \otimes d_l - \sum_a v^a \otimes \phi(e_a)\big).$$

Hence Ψ is a cochain map. Now let $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ be an even solution of (17), and $g \in (S\mathfrak{h}^* \otimes (\wedge\mathfrak{h})^-)_{\mathfrak{h}\text{-inv}}$ an even solution of the corresponding equation for $\mathfrak{h}$.

As we observed in the proof of Proposition 6.1, the two elements

$$\tilde{f}_0 = (\phi^* \otimes 1)g, \;\; \tilde{f}_1 = (1 \otimes \phi)(f) + u$$

of $(S\mathfrak{g}^* \otimes (\wedge\mathfrak{h})^-)_{\mathfrak{g}\text{-inv}}$ both satisfy the equation

$$\partial\tilde{f} + \tfrac{1}{2}[\tilde{f}, \tilde{f}]_{\wedge\mathfrak{h}} = \sum_l \phi^*(q^l) \otimes d_l - \sum_a v^a \otimes \phi(e_a).$$

By Theorem 3.8, it follows that $\tilde{f}_0$ and $\tilde{f}_1$ are gauge equivalent, up to addition of an even element in $(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge^{1-i}\mathfrak{h})_{\mathfrak{h}\text{-inv}}$. The argument at the beginning of Section 4.3 shows that the action by elements in $(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge^{1-i}\mathfrak{h})_{\mathfrak{h}\text{-inv}}$ is trivial, up to homotopy. Hence we can

restrict consideration to gauge equivalent $\tilde{f}_1 = \exp(s_1).\tilde{f}_0$, where $s \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{h})^-)_{\mathfrak{g}\text{-inv}}$ is odd. These then belong to a family $\tilde{f}(t) = \exp(s(t)).\tilde{f}_0$ of solutions, where $s(t) = ts_1$. We have,

$$\frac{d\tilde{f}}{dt} + \partial s + [\tilde{f}, s]_{\wedge\mathfrak{h}} = 0.$$

Each $\tilde{f}(t)$ defines a cochain map

$$\tilde{\Psi}(t) = e^{\iota(\tilde{f}(t))} \circ (\phi^* \otimes 1) \colon (S\mathfrak{h}^*)_{\mathfrak{h}\text{-inv}} \otimes \mathcal{M}_{\mathfrak{h}\text{-inv}} \to (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathfrak{g}\text{-inv}}.$$

Consider the family of operators $H(t) = \tilde{\Psi}(t) \circ \iota(s(t))$. Arguing as in the proof of Proposition 4.5, one shows that

$$\frac{d\tilde{\Psi}}{dt} = H \circ \tilde{\mathrm{d}}_{\mathfrak{h}} + \mathrm{d}_{\mathfrak{g}} \circ H.$$

Hence the maps $\tilde{\Psi}(t)$ are all homotopic, where the homotopy respects the $(S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}}$-module structures. $\square$

Suppose now that $\mathcal{M}$ is a $\mathfrak{g}$-differential *algebra*, that is $\mathcal{M}$ is a $\mathfrak{g}$-differential space which is also a graded algebra, in such a way that $\mathrm{d}, \iota(\xi), L(\xi)$ are all derivations for the product $\mu_{\mathcal{M}} \colon \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$. In this case, the Cartan complex $C_{\mathfrak{g}}(\mathcal{M})$ inherits a product structure for which $\mathrm{d}_{\mathfrak{g}}$ is a derivation. Hence, the product descends to the cohomology $H_{\mathfrak{g}}(\mathcal{M})$.

By contrast, the differential $\tilde{\mathrm{d}}_{\mathfrak{g}}$ for the small Cartan model is not a derivation for the obvious product structure on $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{M}_{\mathrm{inv}}$. Instead, define a new (non-associative) multiplication $\odot$ on $(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \mathcal{M}_{\mathrm{inv}}$ by

$$(p \otimes y) \odot (p' \otimes y') = (1 \otimes \mu_{\mathcal{M}}) e^{\iota(u)} (pp' \otimes y \otimes y'),$$

where $u \in (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge\mathfrak{g} \otimes \wedge\mathfrak{g})_{\mathfrak{g}\text{-inv}}$ is a solution of (47).

Theorem 6.5. *The differential $\tilde{\mathrm{d}}_{\mathfrak{g}}$ on the small Cartan model is a derivation with respect to the product $\odot$,*

$$\tilde{\mathrm{d}}_{\mathfrak{g}}(x \odot x') = \tilde{\mathrm{d}}_{\mathfrak{g}}(x) \odot x' + (-1)^{|x|} x \odot \tilde{\mathrm{d}}_{\mathfrak{g}}(x').$$

The induced product on the equivariant cohomology of $\mathcal{M}$ coincides with that from the usual Cartan model.

Proof. This follows from Theorem 6.4, specialized to the diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$. $\square$

6.3. **Chevalley-Koszul complex.** Suppose $\phi \colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of reductive Lie algebras. The dual map $\phi^* \colon \mathfrak{h}^* \to \mathfrak{g}^*$ extends to a homomorphism of $\mathfrak{g}$-differential algebras

$$\phi^* \colon W\mathfrak{h} \to W\mathfrak{g}.$$

Suppose $\mathcal{M}$ is an $\mathfrak{g}$-differential $W\mathfrak{g}$-module, $\mathcal{N}$ is an $\mathfrak{h}$-differential $W\mathfrak{h}$-module, and $F \colon \mathcal{N} \to \mathcal{M}$ is a homomorphism of $\mathfrak{g}$-differential spaces such that the following diagram commutes:

$$\begin{tikzcd} W\mathfrak{h} \otimes \mathcal{N} \arrow[r] \arrow[d, "\phi^* \otimes F"'] & \mathcal{N} \arrow[d, "F"] \\ W\mathfrak{g} \otimes \mathcal{M} \arrow[r] & \mathcal{M} \end{tikzcd}$$

The geometric setting to have in mind is that of a reduction of the structure group of a principal with connection. The following result may be proved along the same lines as Theorem 6.4:

Theorem 6.6. *Let $u \in (S\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}} \otimes (\wedge\mathfrak{h})_{\mathfrak{g}\text{-inv}}$ be a solution of* (44)*. Then the composition of maps*

$$(F \otimes 1) \circ e^{\iota(u)} \circ (1 \otimes \phi^*) \colon \mathcal{N}_{\mathfrak{h}\text{-basic}} \otimes (\wedge\mathfrak{h}^*)_{\mathfrak{h}\text{-inv}} \to \mathcal{M}_{\mathfrak{g}\text{-basic}} \otimes (\wedge\mathfrak{g}^*)_{\mathfrak{g}\text{-inv}}$$

is a homomorphism of differential $(\wedge\mathfrak{g})_{\mathfrak{h}\text{-inv}}$-modules. Under the isomorphism from Theorem 5.5, this map induces the same map in cohomology as the cochain map $F \colon \mathcal{N}_{\mathfrak{h}\text{-inv}} \to \mathcal{M}_{\mathfrak{g}\text{-inv}}$.

Appendix A. The Halperin complex

Suppose $P \to B$ is a principal G-bundle, F any G-manifold, and $P \times_G F$ the associated bundle with fiber F. An unpublished result of Halperin (quoted in [7, page 569]) describes the de Rham cohomology of $P \times_G F$ as the cohomology of a certain differential on $\Omega(B) \otimes \Omega(F)_{\mathrm{inv}}$. The following is an algebraic generalization of Halperin's result.

Theorem A.1. *Suppose that $\mathcal{N}$ is a $\mathfrak{g}$-differential $W\mathfrak{g}$-module, and that $\mathcal{M}$ is any $\mathfrak{g}$-differential space. Let $\mathcal{N}_{\mathrm{basic}} \otimes \mathcal{M}_{\mathrm{inv}}$ be equipped with the differential,*

$$\mathrm{d}^{\mathcal{N}} \otimes 1 + 1 \otimes \mathrm{d}^{\mathcal{M}} - \sum p^j \otimes \iota^{\mathcal{M}}(c_j). \tag{49}$$

Define a nilpotent operator of degree 0, $\alpha = \sum_a y^a \otimes \iota^{\mathcal{M}}(e_a) \in \mathrm{End}(\mathcal{N} \otimes \mathcal{M})$, and let $f \in (S\mathfrak{g}^* \otimes (\wedge\mathfrak{g})^-)_{\mathrm{inv}}$ be a degree 0 element solving (17)*. Then the map*

$$e^{\alpha} \circ e^{\iota^{\mathcal{M}}(f)} \colon \mathcal{N}_{\mathrm{basic}} \otimes \mathcal{M}_{\mathrm{inv}} \to (\mathcal{N} \otimes \mathcal{M})_{\mathrm{basic}}, \tag{50}$$

is a cochain map. It is a homotopy equivalence of differential $(S\mathfrak{g}^*)_{\mathrm{inv}}$-modules, provided at least one of the $\mathfrak{g}$-differential spaces $\mathcal{N}$, $\mathcal{M}$ is a direct sum of the kernel and image for the action of the Casimir operator $\mathrm{Cas}_{\mathfrak{g}}$.

Proof. The operator $e^{-\alpha}$ on $\mathcal{N} \otimes \mathcal{M}$ takes the contractions on $\mathcal{N} \times \mathcal{M}$ to contractions on the first factor,

$$e^{\alpha} \circ (\iota^{\mathcal{N}}(\xi) + \iota^{\mathcal{M}}(\xi)) \circ e^{-\alpha} = \iota^{\mathcal{N}}(\xi).$$

In particular, it restricts to an isomorphism $(\mathcal{N} \otimes \mathcal{M})_{\mathrm{basic}} \to (\mathcal{N}_{\mathrm{hor}} \otimes \mathcal{M})_{\mathrm{inv}}$. For the induced differential on $(\mathcal{N}_{\mathrm{hor}} \otimes \mathcal{M})_{\mathrm{inv}}$ one finds, after short calculation (cf. [12, page 18])

$$\mathrm{d}' := e^{\alpha} \circ (\mathrm{d}^{\mathcal{N}} + \mathrm{d}^{\mathcal{M}}) \circ e^{-\alpha} = \mathrm{d}^{\mathcal{N}}_{\mathrm{hor}} + \mathrm{d}^{\mathcal{M}} - \sum_a v^a \, \iota^{\mathcal{M}}(e_a).$$

(If $\mathcal{N} = W\mathfrak{g}$, this is Kalkman's proof [12] of the equivalence between $(W\mathfrak{g} \otimes \mathcal{M})_{\mathrm{basic}}$ and the Cartan model $(S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}}$.) The operator $\exp(\iota^{\mathcal{M}}(f))$ commutes with $\iota^{\mathcal{N}}(\xi)$, and therefore preserves $(\mathcal{N}_{\mathrm{hor}} \otimes \mathcal{M})_{\mathrm{inv}}$. It also commutes with $\mathrm{d}^{\mathcal{N}}_{\mathrm{hor}}$ and $\sum_a v^a \, \iota^{\mathcal{M}}(e_a)$, hence the only new contributions arise from the commutator with $\mathrm{d}^{\mathcal{M}}$. We find,

$$\mathrm{d}'' := e^{-\iota^{\mathcal{M}}(f)} \circ \mathrm{d}' \circ e^{\iota^{\mathcal{M}}(f)} = \mathrm{d}^{\mathcal{N}}_{\mathrm{hor}} + \mathrm{d}^{\mathcal{M}} - \sum_j p^j \iota^{\mathcal{M}}(c_j) + \sum_a \iota^{\mathcal{M}}(\iota^*(e^a) f) \circ L^{\mathcal{M}}(e_a). \tag{51}$$

On the subalgebra $\mathcal{N}_{\mathrm{basic}} \otimes \mathcal{M}_{\mathrm{inv}} \subset (\mathcal{N}_{\mathrm{hor}} \otimes \mathcal{M})_{\mathrm{inv}}$, the differential simplifies to (49), as desired. Working backwards, this defines the cochain map (50). To show that (50) is a homotopy equivalence, it suffices to show that the inclusion of $\mathcal{N}_{\mathrm{basic}} \otimes \mathcal{M}_{\mathrm{inv}}$ into $(\mathcal{N}_{\mathrm{hor}} \otimes \mathcal{M})_{\mathrm{inv}}$ is a homotopy equivalence. This is done by a straightforward extension of the argument given in the proof of Theorem 4.2. The only fact needed is that $(\mathcal{N}_{\mathrm{hor}} \otimes \mathcal{M})_{\mathrm{inv}}$ is a direct sum of the kernel and image of $\mathcal{L}_0 = \mathrm{Cas}^{\mathcal{N}}_{\mathfrak{g}} \otimes 1$, which follows by assumption. (Note that $\mathcal{L}_0 = 1 \otimes \mathrm{Cas}^{\mathcal{M}}_{\mathfrak{g}}$ on invariants.) ∎

Note that Theorem A.1 specializes to Theorem 4.2 for $\mathcal{N} = W\mathfrak{g}$. Similarly, it contains part of Theorem 5.5 as the special case $\mathcal{M} = \wedge\mathfrak{g}^*$.

Appendix B. Koszul duality

In this appendix, we discuss the Koszul duality between differential graded modules over symmetric and exterior algebras. Much of this discussion is already implicit in Koszul's work [15], and has appeared in the literature in various degrees of generality [6, 7, 10, 11]. For simplicity, all differential complexes in this appendix are assumed to be bounded below, i.e. equal to 0 in sufficiently negative degrees.

Let $\mathcal{P}$ be any finite-dimensional graded vector space, concentrated in odd negative degrees, and let $\mathcal{P}^*$ be its dual space with grading $(\mathcal{P}^*)^i = (\mathcal{P}^{-i})^*$. Write $\tilde{\mathcal{P}} = \mathcal{P}[-1]$ and $\mathcal{P}^* = \mathcal{P}^*[1]$.

Fix dual bases c_j, c^j of $\mathcal{P}, \mathcal{P}^*$, and let p_j, p^j denote the corresponding dual bases of $\tilde{\mathcal{P}}, \tilde{\mathcal{P}}^*$. The Koszul complex of $\mathcal{P}$ is the differential graded algebra

$$K(\mathcal{P}) = S\tilde{\mathcal{P}}^* \otimes \wedge\mathcal{P}^*, \quad \mathrm{d}_{K(\mathcal{P})} = \sum_j p^j \otimes \iota(c_j).$$

As is well-known, $K(\mathcal{P})$ is acyclic. Let $K(\mathcal{P})^-$ denote the Koszul complex with differential $\mathrm{d}_{K(\mathcal{P})^-} = -\mathrm{d}_{K(\mathcal{P})}$.

A *differential $S\tilde{\mathcal{P}}^*$-module* is a differential space (X, d_X), which is also an $S\tilde{\mathcal{P}}^*$-module, in such a way that the action of any element in $S\tilde{\mathcal{P}}^*$ commutes with the differential. Abusing notation, we will denote the action of $p^j \in S\tilde{\mathcal{P}}^*$ on X simply by p^j. If X_1, X_2 are two differential $S\tilde{\mathcal{P}}^*$-modules, their tensor product $X_1 \otimes X_2$ is again a differential $S\tilde{\mathcal{P}}^*$-module.

In a similar way, we define *differential $\wedge\mathcal{P}$-modules* (Y, d_Y). We will denote the action of $c_j \in \mathcal{P}$ by $\iota(c_j)$. Given two differential $\wedge\mathcal{P}$-modules, their tensor product is again a differential $\wedge\mathcal{P}$-module.

Note that $K(\mathcal{P})$ is both a differential $S\tilde{\mathcal{P}}^*$-module and a differential $\wedge\mathcal{P}$-module. The augmentation map $K(\mathcal{P}) \to \mathbb{F}$ is a homomorphism of differential $S\tilde{\mathcal{P}}^*$-modules, while the coaugmentation map $\mathbb{F} \to K(\mathcal{P})$ is a homomorphism of differential $\wedge\mathcal{P}$-modules.

There is a covariant functor h from the category of differential $S\tilde{\mathcal{P}}^*$-modules to the category of differential $\wedge\mathcal{P}$-modules, taking X to

$$hX = X \otimes \wedge\mathcal{P}^*, \quad \mathrm{d}_{hX} = \mathrm{d}_X \otimes 1 + \sum_j p^j \otimes \iota(c_j),$$

and taking a morphism $\phi\colon X \to X'$ to $h\phi = \phi \otimes 1\colon hX \to hX'$. The filtration of hX coming from the grading on $\wedge\mathcal{P}^*$ defines a spectral sequence for hX, with E_2-term $H(X) \otimes \wedge\mathcal{P}^*$. In particular, the functor h preserves quasi-isomorphisms.

Similarly, there is a covariant functor t from the category of differential $\wedge\mathcal{P}$-modules to the category of differential $S\tilde{\mathcal{P}}^*$-modules, taking Y to

$$tY = S\tilde{\mathcal{P}}^* \otimes Y, \quad \mathrm{d}_{tY} = 1 \otimes \mathrm{d}_Y - \sum_j p^j \otimes \iota(c_j).$$

and taking a morphism $\psi\colon Y \to Y'$ to $t\psi = 1 \otimes \psi\colon tY \to tY'$. Again, t preserves quasi-isomorphisms.

Theorem B.1 (Koszul duality)**.** (a) *For any differential $\wedge\mathcal{P}$-module Y there is a canonical isomorphism of differential $\wedge\mathcal{P}$-modules,*

$$htY \cong K(\mathcal{P}) \otimes Y,$$

and hence a canonical quasi-isomorphism $Y \to htY$.

(b) *For any differential $S\tilde{\mathcal{P}}^*$-module X, there is a canonical isomorphism of differential $S\tilde{\mathcal{P}}^*$-modules,*

$$thX \cong K(\mathcal{P})^- \otimes X$$

and hence a canonical quasi-isomorphism $thX \to X$.

Proof. By definition, the differential $\wedge\mathcal{P}$-module htY is equal to $K(\mathcal{P}) \otimes Y$ as a vector space, but with $\wedge\mathcal{P}$-module structure and differential given by

$$\begin{aligned}\iota_{htY}(c_j) &= \iota_{K(\mathcal{P})}(c_j) \otimes 1,\\ \mathrm{d}_{htY} &= \mathrm{d}_{K(\mathcal{P})} \otimes 1 + 1 \otimes \mathrm{d}_Y - \sum_j p^j \otimes \iota_Y(c_j).\end{aligned}$$

The endomorphism $\alpha = \sum_k c^k \otimes \iota_Y(c_k)$ of $K(\mathcal{P}) \otimes Y$ is nilpotent and has degree 0. Hence, its exponential is a well-defined automorphism of degree 0. A straightforward calculation, using $\mathrm{Ad}(e^{-\alpha}) = e^{-\operatorname{ad}(\alpha)}$, shows

$$\begin{aligned}\mathrm{Ad}(e^{-\alpha})(\iota_{htY}(c_j)) &= \iota_{K(\mathcal{P})}(c_j) \otimes 1 + 1 \otimes \iota_Y(c_j),\\ \mathrm{Ad}(e^{-\alpha})(\mathrm{d}_{htY}) &= \mathrm{d}_{K(\mathcal{P})} \otimes 1 + 1 \otimes \mathrm{d}_Y.\end{aligned}$$

It follows that $e^\alpha \colon K(\mathcal{P}) \otimes Y \to htY$ is an isomorphism of differential $\wedge\mathcal{P}$-modules. Similarly, $thX = K(\mathcal{P}) \otimes X$ as a vector space, but with $S\tilde{\mathcal{P}}^*$-module structure and differential given by

$$\begin{aligned}\iota_{thX}(p^j) &= p^j \otimes 1,\\ \mathrm{d}_{thX} &= -\mathrm{d}_{K(\mathcal{P})} \otimes 1 + 1 \otimes \mathrm{d}_X - \sum_j \iota_{K(Y)}(c_j) \otimes p^j.\end{aligned}$$

The endomorphism $\beta = \sum_j \iota_S(p_j) \otimes p^j$ of $K(\mathcal{P}) \otimes X$ is nilpotent and has degree 0. We find,

$$\begin{aligned}\mathrm{Ad}(e^\beta)\iota_{thX}(p^j) &= p^j \otimes 1 + 1 \otimes p^j,\\ \mathrm{Ad}(e^\beta)\mathrm{d}_{thX} &= -\mathrm{d}_{K(\mathcal{P})} \otimes 1 + 1 \otimes \mathrm{d}_X.\end{aligned}$$

Hence $e^{-\beta}$ gives an isomorphism of differential $S\tilde{\mathcal{P}}^*$-modules, $K(\mathcal{P})^- \otimes X \to thX$. □

We are now in position to explain the Koszul duality between Theorems 4.2 and 5.5. Suppose $\mathcal{M}$ is any $\mathfrak{g}$-differential space. As before, we denote by $\mathcal{P}$ the primitive subspace of $(\wedge\mathfrak{g})_{\mathrm{inv}}$. Then the subspace $\mathcal{M}_{\mathrm{inv}}$ of invariants is a differential $\wedge\mathcal{P}$-module, while the Cartan model $(S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}}$ is a differential $S\tilde{\mathcal{P}}^*$-module. Applying the functor t to $\mathcal{M}_{\mathrm{inv}}$, we obtain the small Cartan model,

$$t\mathcal{M}_{\mathrm{inv}} = S\tilde{\mathcal{P}}^* \otimes \mathcal{M}_{\mathrm{inv}}. \tag{52}$$

On the other hand, recall [2] that the map $(W\mathfrak{g} \otimes \mathcal{M})_{\mathrm{basic}} \to (S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}} = C_{\mathfrak{g}}(\mathcal{M})$ induced by the projection $W\mathfrak{g} \to S\mathfrak{g}^*$ is an isomorphism of differential $S\tilde{\mathcal{P}}$-modules. Hence, applying the functor h to the (big) Cartan model we obtain

$$h(S\mathfrak{g}^* \otimes \mathcal{M})_{\mathrm{inv}} = (W\mathfrak{g} \otimes \mathcal{M})_{\mathrm{basic}} \otimes \wedge\mathcal{P}^*. \tag{53}$$

Let $\sim$ denote the relation of quasi-isomorphism in the category of differential $S\tilde{\mathcal{P}}^*$-modules, respectively of differential $\wedge\mathcal{P}$-modules. Since $W\mathfrak{g}$ is acyclic, $(W\mathfrak{g}\otimes\mathcal{M})_{\mathrm{inv}}\sim\mathcal{M}_{\mathrm{inv}}$. The quasi-isomorphism from Theorem 5.5,

$$h(S\mathfrak{g}^*\otimes\mathcal{M})_{\mathrm{inv}}=h(W\mathfrak{g}\otimes\mathcal{M})_{\mathrm{basic}}\sim(W\mathfrak{g}\otimes\mathcal{M})_{\mathrm{inv}}\sim\mathcal{M}_{\mathrm{inv}}$$

implies, by Koszul duality, a quasi-isomorphism of differential $S\tilde{\mathcal{P}}^*$-modules,

$$(S\mathfrak{g}^*\otimes\mathcal{M})_{\mathrm{inv}}\sim th(S\mathfrak{g}^*\otimes\mathcal{M})_{\mathrm{inv}})\sim t\mathcal{M}_{\mathrm{inv}}$$

which is the equivalence of the two Cartan models. Conversely, suppose $\mathcal{N}$ is a $\mathfrak{g}$-differential $W\mathfrak{g}$-module. We have $(S\mathfrak{g}^*\otimes\mathcal{N})_{\mathrm{inv}}\sim\mathcal{N}_{\mathrm{basic}}$ by Cartan's theorem 5.2. Therefore, the quasi-isomorphism from Theorem 4.2

$$\mathcal{N}_{\mathrm{basic}}\sim(S\mathfrak{g}^*\otimes\mathcal{N})_{\mathrm{inv}}\sim t\mathcal{N}_{\mathrm{inv}}$$

yields a quasi-isomorphism of differential $\wedge\mathcal{P}$-modules, as in Theorem 5.5,

$$h\mathcal{N}_{\mathrm{basic}}\sim ht\mathcal{N}_{\mathrm{inv}}\sim\mathcal{N}_{\mathrm{inv}}.$$

References

1. C. Allday and V. Puppe, *On a conjecture of Goresky, Kottwitz and MacPherson*, Canad. J. Math. **51** (1999), no. 1, 3–9. MR **2000e:**18008
2. H. Cartan, *La transgression dans un groupe de Lie et dans un fibré principal*, Colloque de topologie (espaces fibrés) (Bruxelles), Centre belge de recherches mathématiques, Georges Thone, Liège, Masson et Cie., Paris, 1950, pp. 73–81.
3. ______, *Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie.*, Colloque de topologie (espaces fibrés) (Bruxelles), Georges Thone, Liège, Masson et Cie., Paris, 1950.
4. M. Franz, *Koszul duality and equivariant cohomology*, arXiv:math.AT/0307115.
5. ______, *Koszul duality and equivariant cohomology for tori*, Int. Math. Res. Not. (2003), no. 42, 2255–2303. MR 1 997 219
6. M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83.
7. W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology*, Academic Press, New York, 1976, Volume III: Cohomology of principal bundles and homogeneous spaces.
8. V. Gugenheim and L. Lambe, *Perturbation theory in differential homological algebra. I*, Illinois J. Math. **33** (1989), no. 4, 566–582. MR **MR1007895 (91e:**55023)
9. V. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer Verlag, 1999.
10. J. Huebschmann, *Homological perturbations, equivariant cohomology, and Koszul duality*, arXiv:math.AT/0401160.
11. ______, *Relative homological algebra, homological perturbations, equivariant de Rham theory, and Koszul duality*, arXiv:math.DG/0401161.
12. J. Kalkman, *A BRST model applied to symplectic geometry*, Ph.D. thesis, Universiteit Utrecht, 1993.
13. B. Kostant, *Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition* $\mathrm{C}(\mathfrak{g})=\mathrm{End}\,V_\rho\otimes\mathrm{C}(\mathrm{P})$*, and the $\mathfrak{g}$-module structure of* $\bigwedge\mathfrak{g}$, Adv. in Math. **125** (1997), no. 2, 275–350.
14. J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127. MR 12,120g
15. J. L. Koszul, *Sur un type d'algèbres différentielles en rapport avec la transgression*, Colloque de topologie (espaces fibrés), Bruxelles, 1950, Georges Thone, Liège, 1951, pp. 73–81. MR 13,109a
16. ______, *Selected papers of J. L. Koszul*, Series in Pure Mathematics, vol. 17, World Scientific, Singapore, 1994.
17. T. Maszczyk and A. Weber, *Koszul duality for modules over Lie algebras*, Duke Math. J. **112** (2002), no. 3, 511–520.

18. L. Nicolaescu, *On a theorem of Henri Cartan concerning the equivariant cohomology*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **45** (1999), no. 1, 17–38 (2000), see also arXiv:math.DG/0005068.

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